Perspective Projection

CS 543 / ECE 549 – Saurabh Gupta
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http://saurabhg.web.illinois.edu/teaching/ece549/sp2020/

Many slides adapted from S. Seitz, L. Lazebnik, B. Hariharan.
Overview of next two lectures

• The pinhole projection model
  • Geometric properties
  • Perspective projection matrix

• Cameras with lenses
  • Depth of focus
  • Field of view
  • Lens aberrations

• Digital sensors
Let’s design a camera

Idea 1: put a piece of film in front of an object
Do we get a reasonable image?
Pinhole camera

Add a barrier to block off most of the rays
Pinhole camera

- Captures pencil of rays – all rays through a single point: aperture, center of projection, optical center, focal point, camera center
- The image is formed on the image plane
Pinhole cameras are everywhere

Tree shadow during a solar eclipse
photo credit: Nils van der Burg
http://www.physicstogo.org/index.cfm

Slide by Steve Seitz
Camera obscura

- Basic principle known to Mozi (470-390 BCE), Aristotle (384-322 BCE)
- Drawing aid for artists: described by Leonardo da Vinci (1452-1519)

Gemma Frisius, 1558
Turning a room into a camera obscura

After scouting rooms and reserving one for at least a day, Morell masks the windows except for the aperture. He controls three elements: the size of the hole, with a smaller one yielding a sharper but dimmer image; the length of the exposure, usually eight hours; and the distance from the hole to the surface on which the outside image falls and which he will photograph. He used 4 x 5 and 8 x 10 view cameras and lenses ranging from 75 to 150 mm.

After he’s done inside, it gets harder. “I leave the room and I am constantly checking the weather, I’m hoping the maid reads my note not to come in, I’m worrying that the sun will hit the plastic masking and it will fall down, or that I didn’t trigger the lens.”

From *Grand Images Through a Tiny Opening*, *Photo District News*, February 2005

Camera Obscura: View of Central Park
Looking West in Bedroom. Summer, 2018

http://www.abelardomorell.net/project/camera-obscura/

Adapted from L. Lazebnik.
Turning a room into a camera obscura

My hotel room, contrast enhanced.

The view from my window

Accidental pinholes produce images that are unnoticed or misinterpreted as shadows

A. Torralba and W. Freeman, Accidental Pinhole and Pinspeck Cameras, CVPR 2012

Slide by L. Lazebnik.
Pinhole projection model

- To compute the projection $p$ of a scene point $P$, form the visual ray connecting $P$ to the camera center $O$ and find where it intersects the image plane.
Pinhole projection model

- The coordinate system
  - The optical center (O) is at the origin
  - The image plane is parallel to xy-plane or perpendicular to the z-axis, which is the optical axis
Pinhole projection model

Projection equations

- Derived using similar triangles

\[(X, Y, Z) \rightarrow \left(\frac{fX}{Z}, \frac{fY}{Z}\right) = (x, y)\]

Note: instead of dealing with an image that is upside down, most of the time we will pretend that the image plane is in front of the camera center.
Dimensionality reduction: from 3D to 2D

3D world

2D image

Point of observation

What properties of the world are preserved?
- Straight lines, incidence

What properties are not preserved?
- Angles, lengths
Properties of projection

Who is taller?

Which is closer?
Properties of projection

Parallel?

Perpendicular?
Parallel lines converge at a point

\[ L_1(\lambda) = A + \lambda D \]
\[ = (A_X + \lambda D_X, A_Y + \lambda D_Y, A_Z + \lambda D_Z). \]

\[ l_1(\lambda) = \left( f \frac{A_X + \lambda D_X}{A_Z + \lambda D_Z}, f \frac{A_Y + \lambda D_Y}{A_Z + \lambda D_Z} \right) \]

Point \( P_i \)
\[ A + \lambda_i D \]

Adapted from B. Hariharan.
Parallel lines converge at a point

\[ L_1(\lambda) = A + \lambda D \]
\[ = (A_X + \lambda D_X, A_Y + \lambda D_Y, A_Z + \lambda D_Z) \]

\[ l_1(\lambda) = \left( f \frac{A_X + \lambda D_X}{A_Z + \lambda D_Z}, f \frac{A_Y + \lambda D_Y}{A_Z + \lambda D_Z} \right). \]

Study, behavior of \( l_1(\lambda) \) as \( A_Z + \lambda D_Z \to \infty \),
which is same as \( \lambda \to \infty \).

\[ \lim_{\lambda \to \infty} f \frac{A_X + \lambda D_X}{A_Z + \lambda D_Z} = \lim_{\lambda \to \infty} f \frac{A_X/\lambda + D_X}{A_Z/\lambda + D_Z} = \frac{f D_X}{D_Z}. \]

\[ \lim_{\lambda \to \infty} l_1(\lambda) = \left( f \frac{D_X}{D_Z}, f \frac{D_Y}{D_Z} \right). \]

But, what happens if \( D_Z = 0 \)?

Adapted from B. Hariharan.
Parallel lines converge at a point

$D_Z = 0$ lines?

Adapted from B. Hariharan.
Farther away objects are smaller

- Head: \((X, Y+h, Z)\)
- Foot: \((X, Y, Z)\)

Image of foot: \(\left(\frac{fX}{Z}, \frac{fY}{Z}\right)\)
Image of head: \(\left(\frac{fX}{Z}, \frac{f(Y+h)}{Z}\right)\)

Height: \(\frac{f(Y+h)}{Z} - \frac{fY}{Z} = \frac{fh}{Z}\)

Adapted from B. Hariharan.
What about planes?

Plane: \( N_X X + N_Y Y + N_Z Z = c \)

Plane: \( N_X X + N_Y Y + N_Z Z = d \)

\[
\begin{align*}
N_X X + N_Y Y + N_Z Z &= d \\
\frac{N_X f X}{Z} + \frac{N_Y f Y}{Z} + f N_Z &= \frac{f d}{Z} \\
N_X x + N_Y y + f N_Z &= \frac{f d}{Z}
\end{align*}
\]

As \( Z \to \infty \),
\[
N_X x + N_Y y + f N_Z = 0
\]

Planes vanish into a line.

Parallel planes vanish into the same line.

Adapted from B. Hariharan.
Except?

\[ N_X = 0 \text{ and } N_Y = 0. \]

Fronto-parallel plane.
Fronto-parallel planes

- What happens to the projection of a pattern on a plane parallel to the image plane?
  - All points on that plane are at a fixed depth $z$
  - The pattern gets scaled by a factor of $f/z$, but angles and ratios of lengths/areas are preserved

$$(X, Y, Z) \rightarrow \left( \frac{fX}{Z}, \frac{fY}{Z} \right)$$
Horizon: Vanishing line of ground plane

\[ N_X x + N_Y y + f N_Z = 0 \]
Vanishing lines of planes

Is the parachutist above or below the camera?
Comparing heights

Vanishing Point

Slide by Steve Seitz
Measuring height

What is the height of the camera?
Horizon: Vanishing line of ground plane

- **Horizon**: vanishing line of the ground plane
  - All points at the same height as the camera project to the horizon
  - Points higher (resp. lower) than the camera project above (resp. below) the horizon
  - Provides way of comparing height of objects

Adapted from Steve Seitz
Fun with Projective Geometry

Illusion Credit: RN Shepard, Mind Sights: Original Visual Illusions, Ambiguities, and other Anomalies
Perspective cues in art

Masaccio, *Trinity*, Santa Maria Novella, Florence, 1425-28

One of the first consistent uses of perspective in Western art

Slide from L. Lazebnik
Perspective distortion

- What is the shape of the projection of a sphere?
Perspective distortion

• Are the widths of the projected columns equal?
  • The exterior columns are wider
  • This is not an optical illusion, and is not due to lens flaws
  • Phenomenon pointed out by Da Vinci

Source: F. Durand
Perspective distortion: People
Modeling projection

Projection equation: \[(X, Y, Z) \rightarrow \left(\frac{fX}{Z}, \frac{fY}{Z}\right) = (x, y)\]

Note: instead of dealing with an image that is upside down, most of the time we will pretend that the image plane is in front of the camera center.

Source: J. Ponce, S. Seitz
Homogeneous coordinates

$$(X, Y, Z) \rightarrow \left(\frac{fX}{Z}, \frac{fY}{Z}\right)$$

Is this a linear transformation?

- no—division by $z$ is nonlinear

Trick: add one more coordinate:

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad (x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

homogeneous image coordinates

homogeneous scene coordinates

Converting *from* homogeneous coordinates

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w) \quad \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$
Perspective Projection Matrix

Projection is a matrix multiplication using homogeneous coordinates

\[
\begin{bmatrix}
  x \\
y \\
z \\
1
\end{bmatrix}
= \begin{bmatrix}
  \frac{x}{z} \\
\frac{y}{z}
\end{bmatrix}
\Rightarrow \left( f \frac{x}{z}, f \frac{y}{z} \right)
\]
divide by the third coordinate

In practice: lots of coordinate transformations...

\[
\begin{bmatrix}
  2D \\
  \text{point} \\
  (3x1)
\end{bmatrix}
= \begin{bmatrix}
  \text{Camera to} \\
  \text{pixel coord.} \\
  \text{trans. matrix} \\
  (3x3)
\end{bmatrix}
\begin{bmatrix}
  \text{Perspective} \\
  \text{projection matrix} \\
  (3x4)
\end{bmatrix}
\begin{bmatrix}
  \text{World to} \\
  \text{camera coord.} \\
  \text{trans. matrix} \\
  (4x4)
\end{bmatrix}
\begin{bmatrix}
  3D \\
  \text{point} \\
  (4x1)
\end{bmatrix}
\]

Slide from L. Lazebnik
Perspective Projection Matrix

Projection is a matrix multiplication using homogeneous coordinates

\[
\begin{bmatrix}
    x \\
    y \\
    z \\
    1
\end{bmatrix}
= 
\begin{bmatrix}
    \quad & \quad & \quad & x \\
    \quad & \quad & \quad & y \\
    \quad & \quad & \quad & z \\
    \quad & \quad & \quad & 1
\end{bmatrix}
\Rightarrow \left( f \frac{x}{z}, f \frac{y}{z} \right)
\]
divide by the third coordinate

In practice: lots of coordinate transformations…

\[
\begin{bmatrix}
    \text{2D point} \\
    \text{pixel coord.} \\
    \text{trans. matrix}
\end{bmatrix}
= 
\begin{bmatrix}
    \text{Camera to} \\
    \text{perspective} \\
    \text{projection matrix}
\end{bmatrix}
\begin{bmatrix}
    \text{World to} \\
    \text{camera coord.} \\
    \text{trans. matrix}
\end{bmatrix}
\begin{bmatrix}
    \text{3D point}
\end{bmatrix}
\]

Slide from L. Lazebnik
Orthographic Projection

Special case of perspective projection

- Distance from center of projection to image plane is infinite
- Also called “parallel projection”
Orthographic Projection

Special case of perspective projection

- Distance from center of projection to image plane is infinite
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Orthographic Projection

Special case of perspective projection

- Distance from center of projection to image plane is infinite
- Also called “parallel projection”

What’s the projection matrix?

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
= \begin{bmatrix}
x \\
y \\
1
\end{bmatrix} \Rightarrow (x, y)
Recap

\[ (X, Y, Z) \rightarrow \left( \frac{fX}{Z}, \frac{fY}{Z} \right) \]

Cartoon. (Drawing by S. Harris; © 1975 The New Yorker Magazine, Inc.)