

Perspective Projection



CS 543 / ECE 549

<http://saurabhg.web.illinois.edu/teaching/ece549/sp2021/>

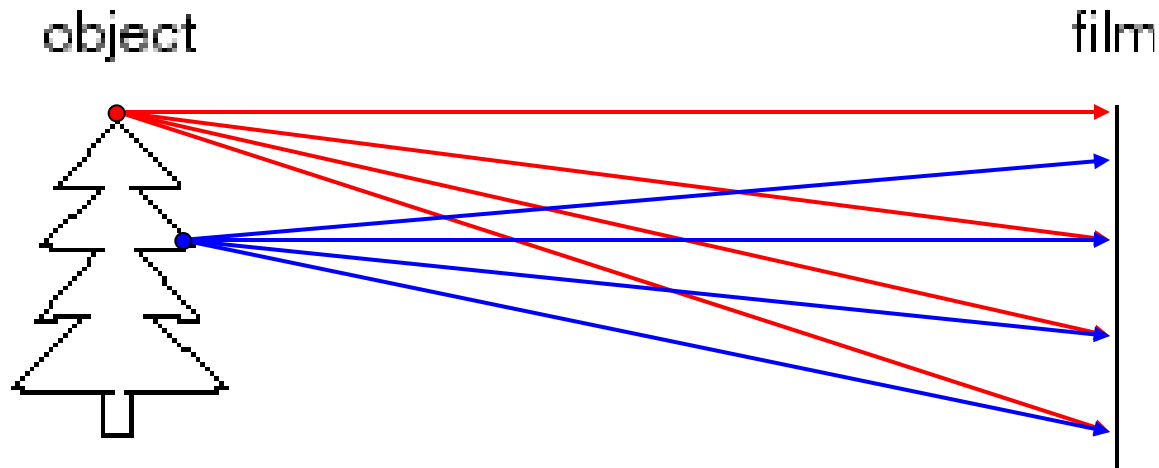
Video source: [Camera Obscura & World of Illusions](#)

Many slides adapted from S. Seitz, L. Lazebnik, B. Hariharan.

Overview of next two lectures

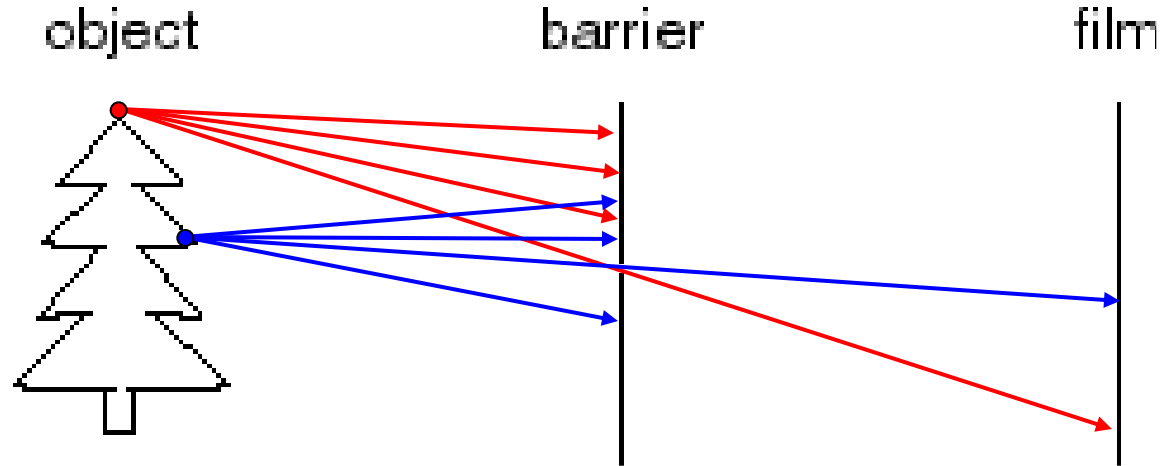
- The pinhole projection model
 - Geometric properties
 - Perspective projection matrix
- Cameras with lenses
 - Depth of focus
 - Field of view
 - Lens aberrations
- Digital sensors

Let's design a camera



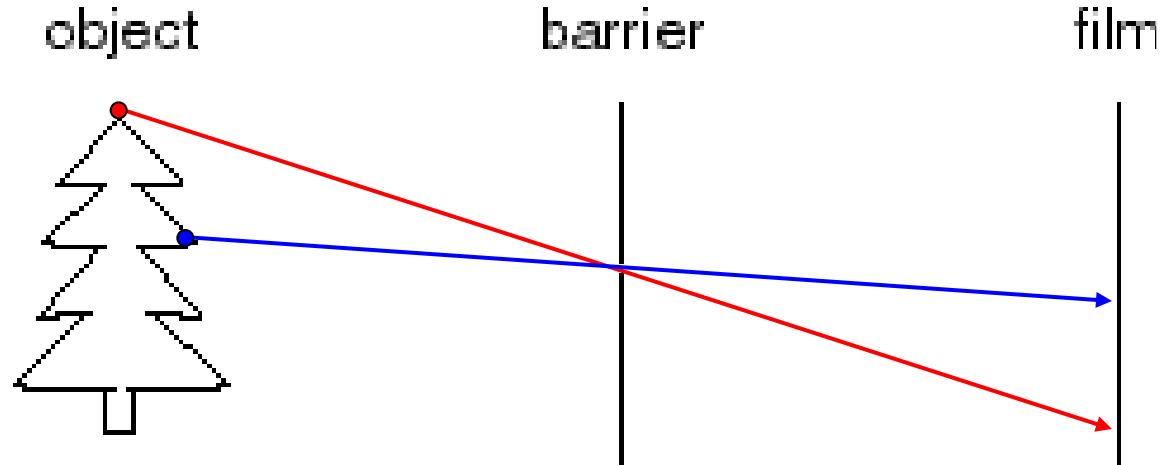
Idea 1: put a piece of film in front of an object
Do we get a reasonable image?

Pinhole camera



Add a barrier to block off most of the rays

Pinhole camera



- Captures **pencil of rays** – all rays through a single point: **aperture, center of projection, optical center, focal point, camera center**
- The image is formed on the **image plane**

Pinhole cameras are everywhere

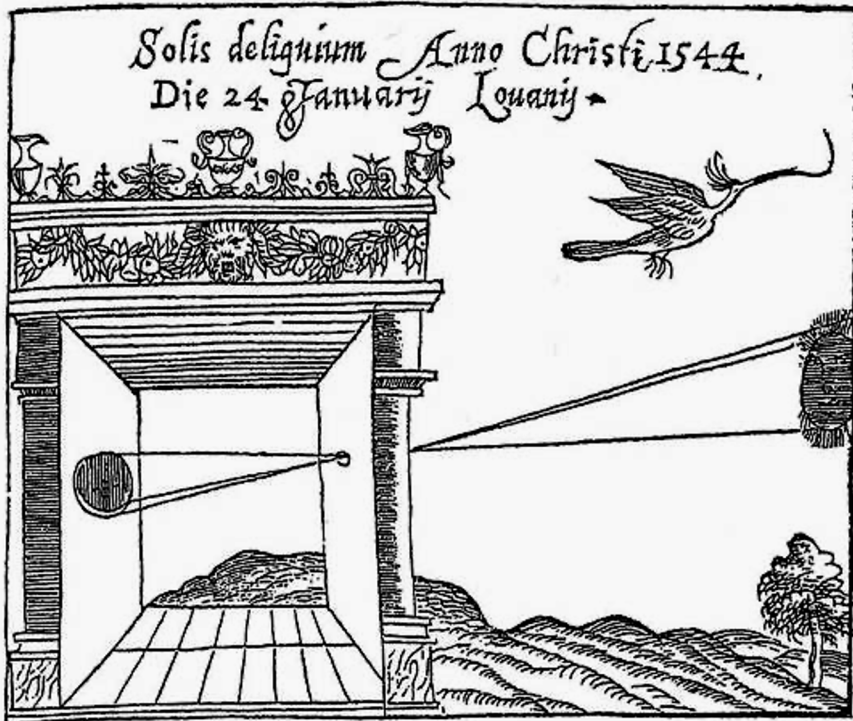


Tree shadow during a solar eclipse

photo credit: Nils van der Burg

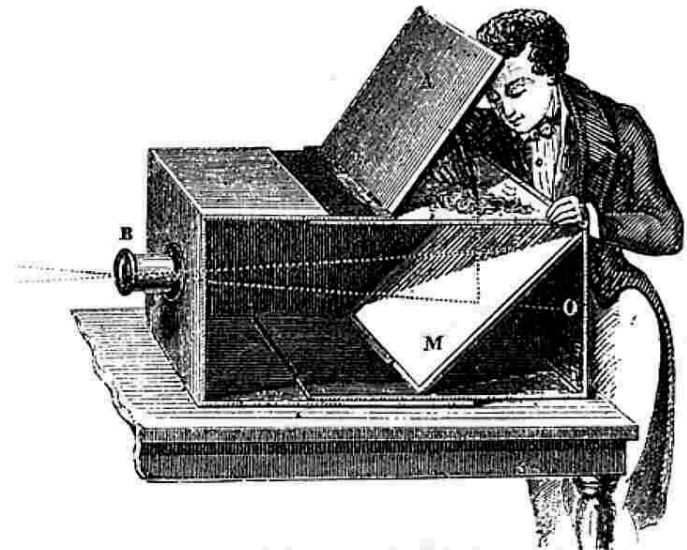
<http://www.physicstogo.org/index.cfm>

Camera obscura



Gemma Frisius, 1558

- Basic principle known to Mozi (470-390 BCE), Aristotle (384-322 BCE)
- Drawing aid for artists: described by Leonardo da Vinci (1452-1519)



Turning a room into a camera obscura



Camera Obscura: View of Central Park
Looking West in Bedroom. Summer, 2018

<http://www.abelardomorell.net/project/camera-obscura/>

After scouting rooms and reserving one for at least a day, Morell masks the windows except for the aperture. He controls three elements: the size of the hole, with a smaller one yielding a sharper but dimmer image; the length of the exposure, usually eight hours; and the distance from the hole to the surface on which the outside image falls and which he will photograph. He used 4 x 5 and 8 x 10 view cameras and lenses ranging from 75 to 150 mm.

After he's done inside, it gets harder. "I leave the room and I am constantly checking the weather, I'm hoping the maid reads my note not to come in, I'm worrying that the sun will hit the plastic masking and it will fall down, or that I didn't trigger the lens."

From *Grand Images Through a Tiny Opening*, **Photo District News**, February 2005

Turning a room into a camera obscura

My hotel room,
contrast enhanced.



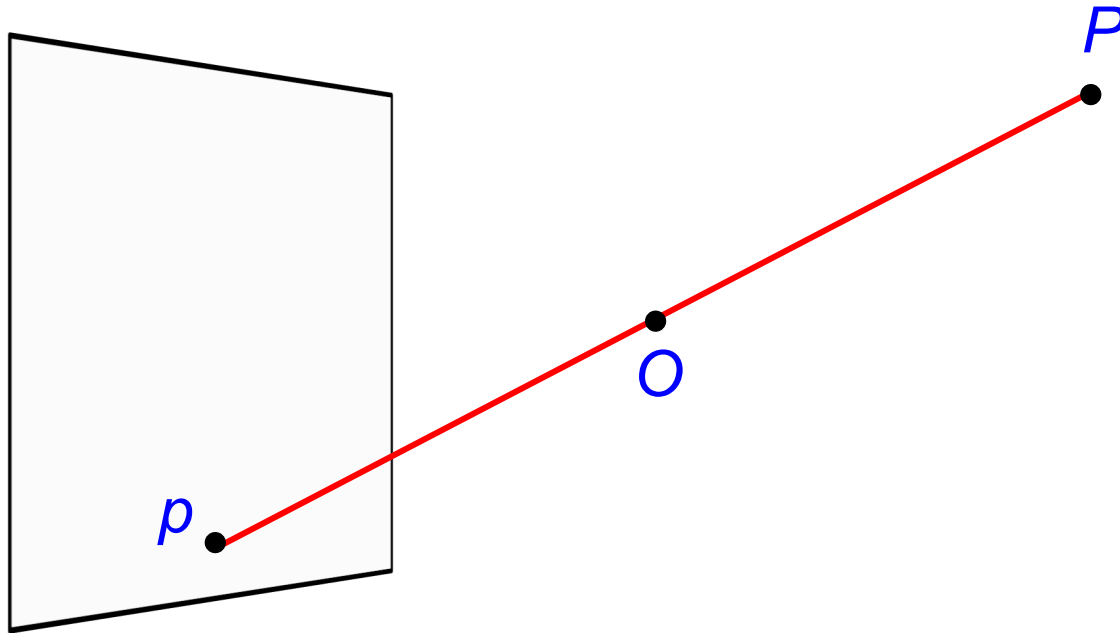
The view from my window



Accidental pinholes produce images that are
unnoticed or misinterpreted as shadows

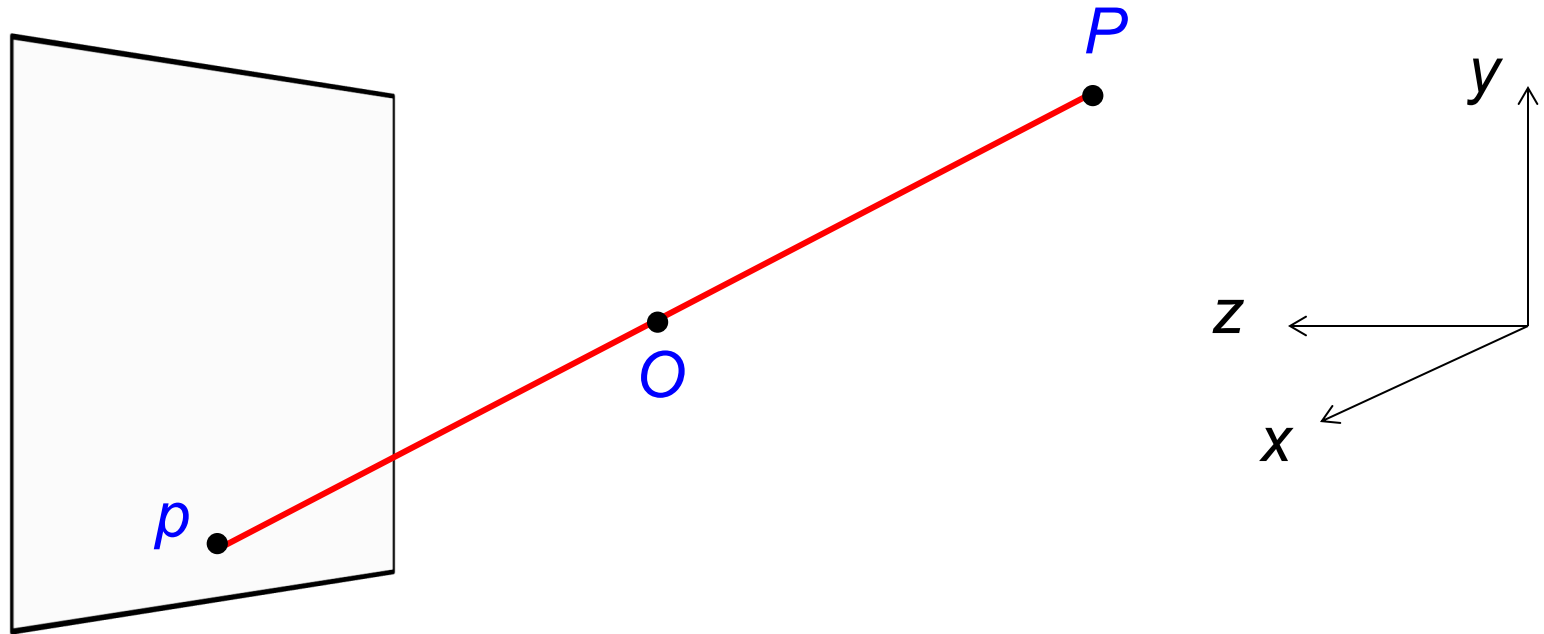
A. Torralba and W. Freeman, [Accidental Pinhole and Pinspeck Cameras](#), CVPR 2012

Pinhole projection model



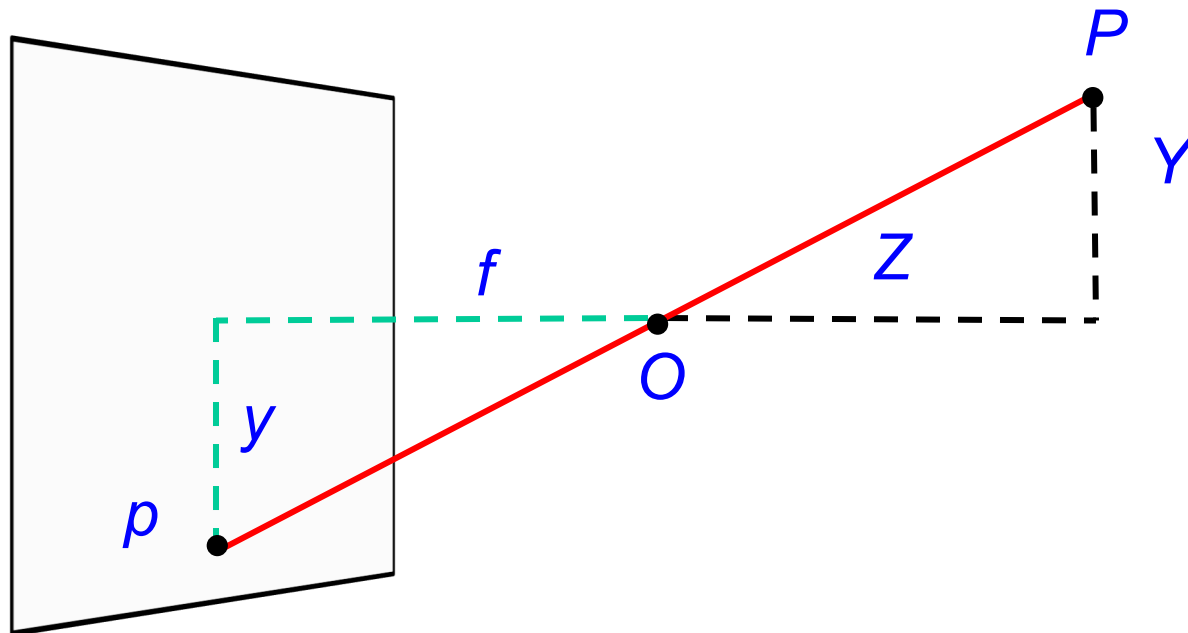
- To compute the projection p of a scene point P , form the **visual ray** connecting P to the **camera center** O and find where it intersects the **image plane**

Pinhole projection model



- The coordinate system
 - The optical center (O) is at the origin
 - The image plane is parallel to xy -plane or perpendicular to the z -axis, which is the *optical axis*

Pinhole projection model



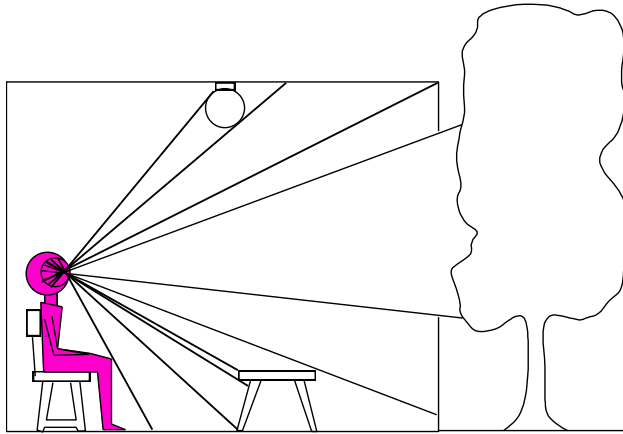
Projection equations

- Derived using similar triangles $(X, Y, Z) \rightarrow \left(\frac{fX}{Z}, \frac{fY}{Z}\right) = (x, y)$

Note: instead of dealing with an image that is upside down, most of the time we will pretend that the image plane is in front of the camera center.

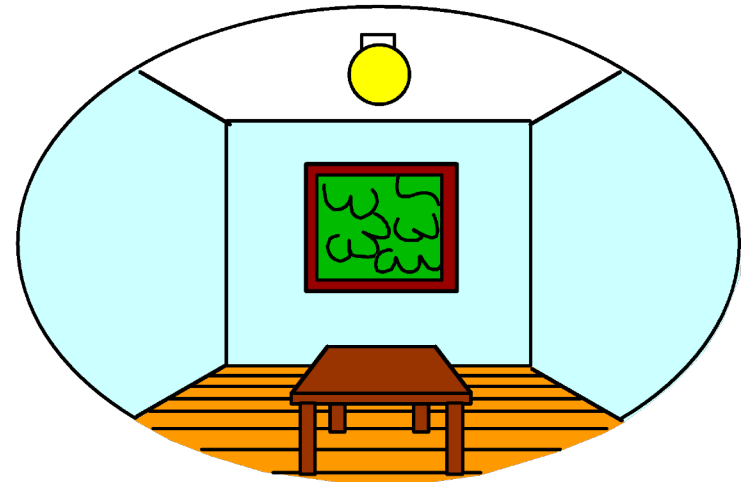
Dimensionality reduction: from 3D to 2D

3D world



Point of observation

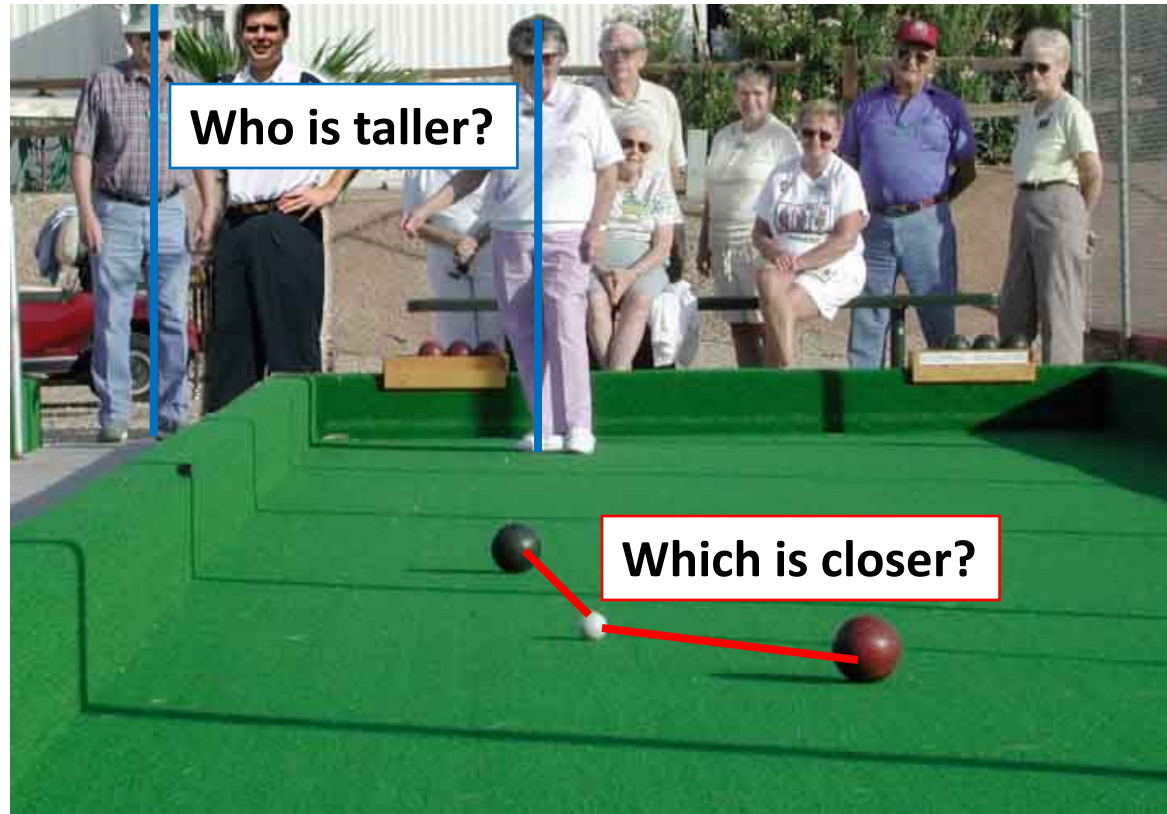
2D image



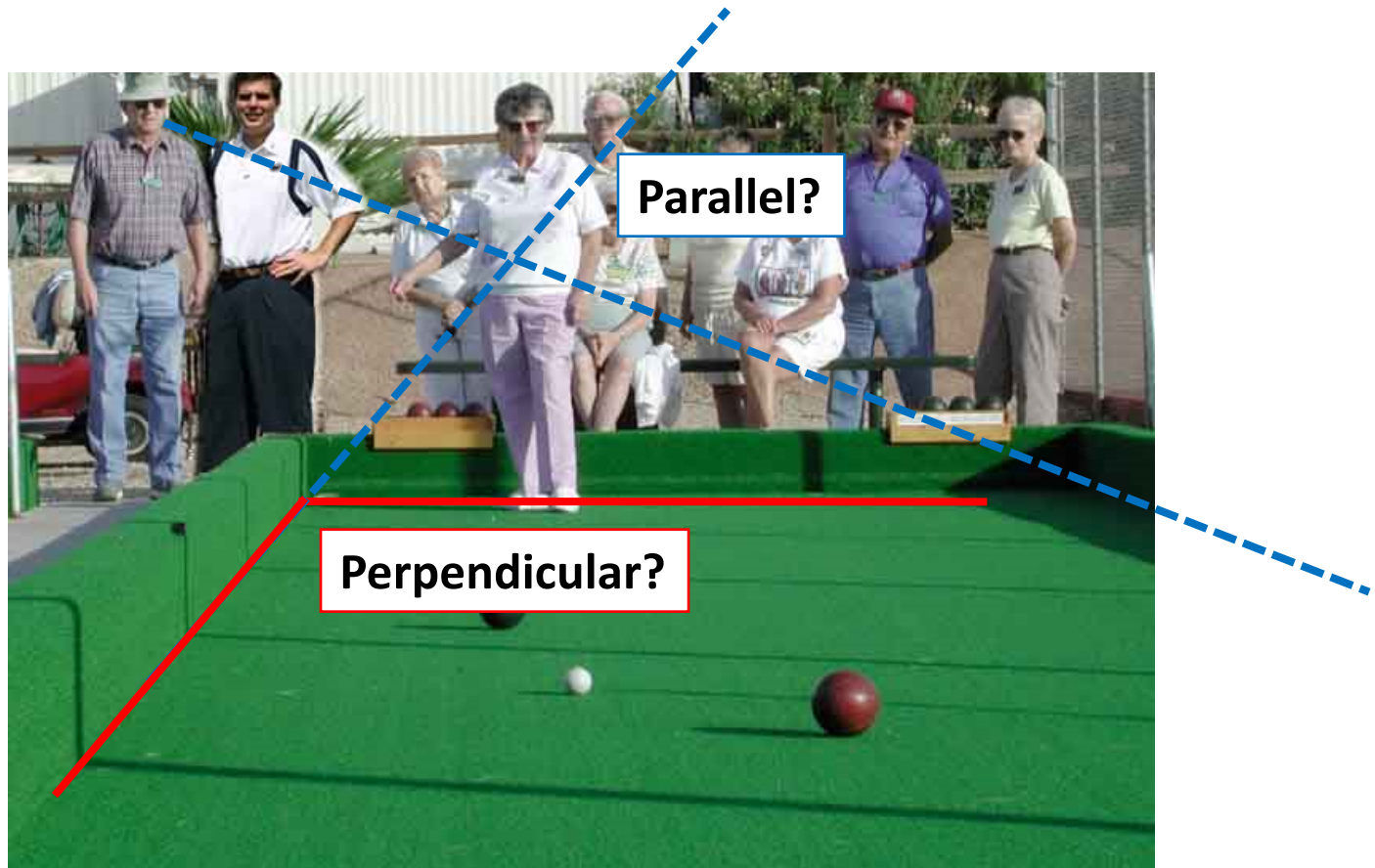
What properties of the world are preserved?

- A. Straight Lines
- B. Angles
- C. Incidence
- D. Lengths
- E. Parallel Lines

Properties of projection



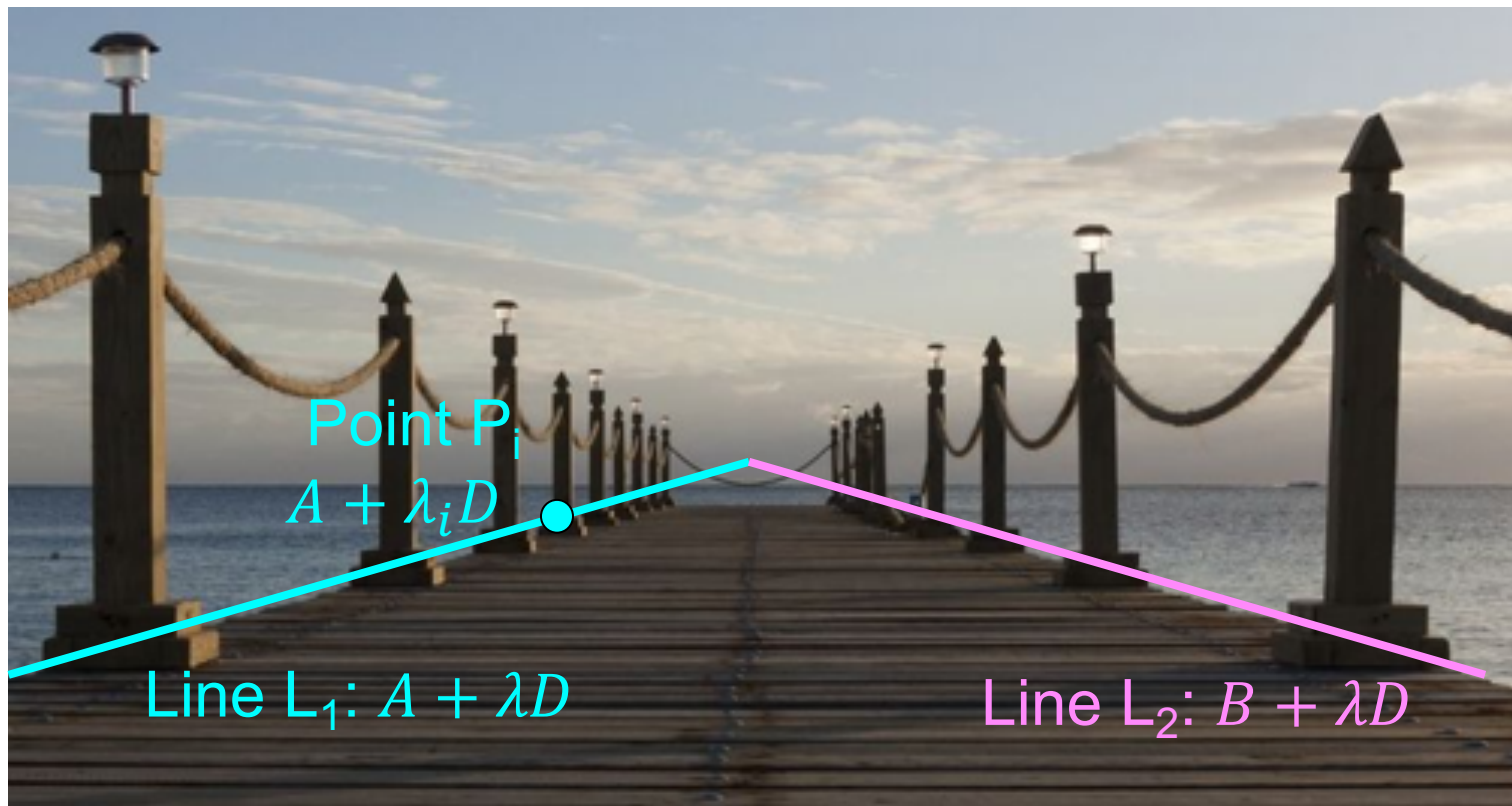
Properties of projection





Vanishing
Point

Parallel lines converge at a point



$$A = (A_X, A_Y, A_Z)$$

$$B = (B_X, B_Y, B_Z)$$

$$D = (D_X, D_Y, D_Z)$$

$$L_1(\lambda) = A + \lambda D$$

$$= (A_X + \lambda D_X, A_Y + \lambda D_Y, A_Z + \lambda D_Z).$$

$$l_1(\lambda) = \left(f \frac{A_X + \lambda D_X}{A_Z + \lambda D_Z}, f \frac{A_Y + \lambda D_Y}{A_Z + \lambda D_Z} \right)$$

Parallel lines converge at a point

$$\begin{aligned}L_1(\lambda) &= A + \lambda D \\ &= (A_X + \lambda D_X, A_Y + \lambda D_Y, A_Z + \lambda D_Z)\end{aligned}$$

$$l_1(\lambda) = \left(f \frac{A_X + \lambda D_X}{A_Z + \lambda D_Z}, f \frac{A_Y + \lambda D_Y}{A_Z + \lambda D_Z} \right).$$

Study, behavior of $l_1(\lambda)$ as $A_Z + \lambda D_Z \rightarrow \infty$,

which is same as $\lambda \rightarrow \infty$.

$$\lim_{\lambda \rightarrow \infty} f \frac{A_X + \lambda D_X}{A_Z + \lambda D_Z} = \lim_{\lambda \rightarrow \infty} f \frac{A_X/\lambda + D_X}{A_Z/\lambda + D_Z} = \frac{f D_X}{D_Z}.$$

$$\lim_{\lambda \rightarrow \infty} l_1(\lambda) = \left(f \frac{D_X}{D_Z}, f \frac{D_Y}{D_Z} \right).$$

But, what happens if $D_Z = 0$?

Parallel lines converge at a point



$D_Z = 0$ lines?

Farther away objects are smaller

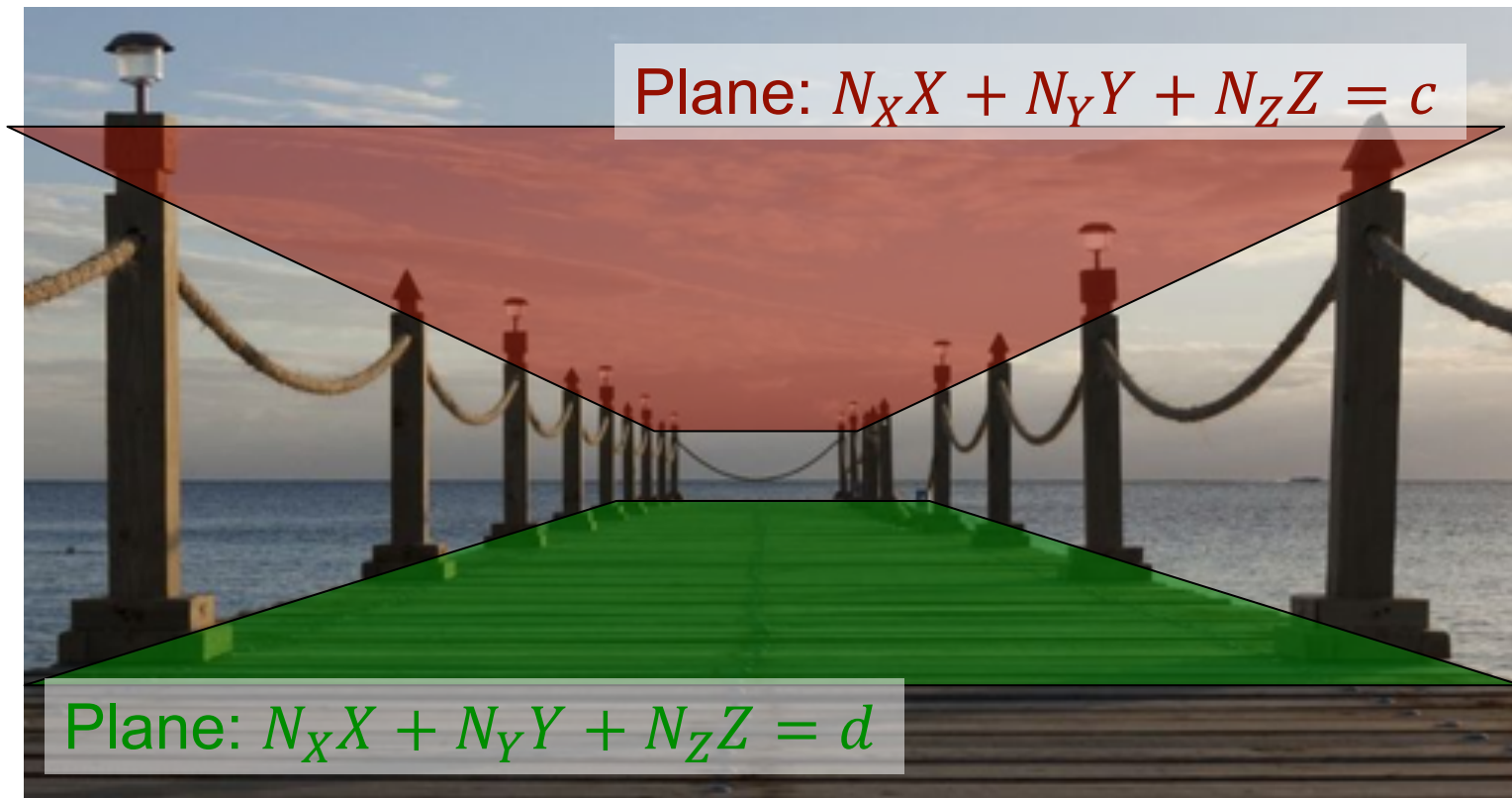


Image of foot: $\left(\frac{fX}{Z}, \frac{fY}{Z}\right)$

Image of head: $\left(\frac{fX}{Z}, \frac{f(Y+h)}{Z}\right)$

$$\text{Height: } \frac{f(Y+h)}{Z} - \frac{fY}{Z} = \frac{fh}{Z}$$

What about planes?



$$N_X X + N_Y Y + N_Z Z = d$$

$$\frac{N_X f X}{Z} + \frac{N_Y f Y}{Z} + f N_Z = \frac{f d}{Z}$$

$$N_X x + N_Y y + f N_Z = \frac{f d}{Z}$$

As $Z \rightarrow \infty$,

$$N_X x + N_Y y + f N_Z = 0$$

Planes vanish into a line.

Parallel planes vanish into the same line.

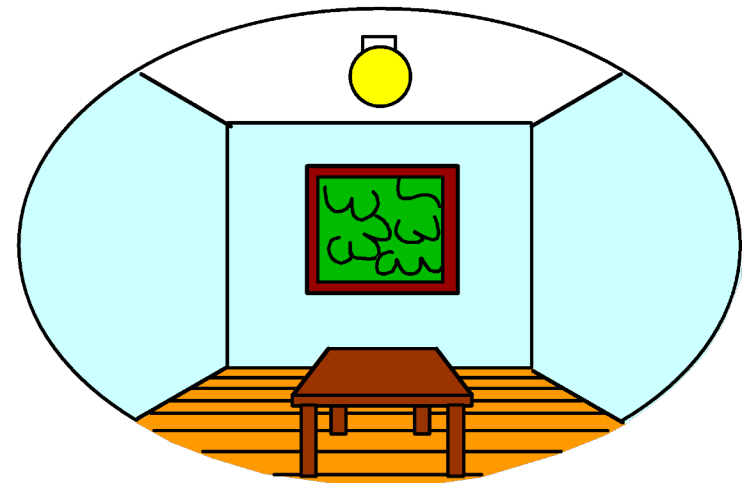
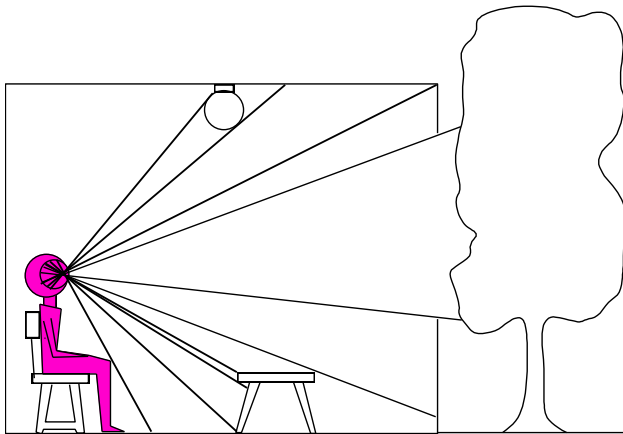
Except?



$N_X = 0$ and $N_Y = 0$.
Fronto-parallel plane.

Fronto-parallel planes

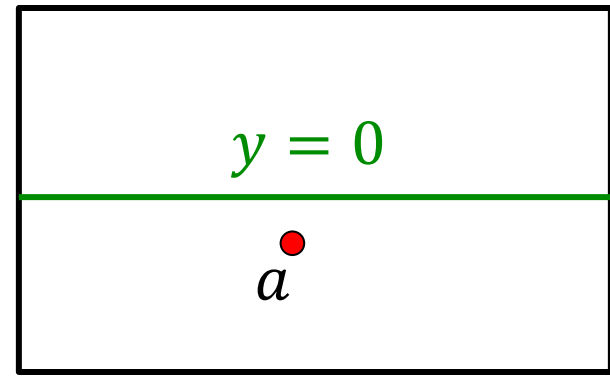
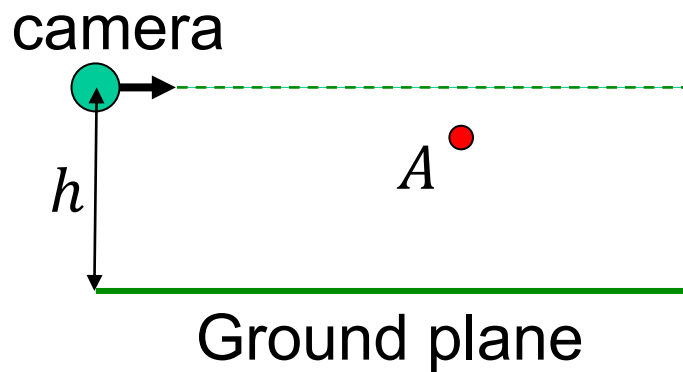
- What happens to the projection of a pattern on a plane parallel to the image plane?
 - All points on that plane are at a fixed *depth* z
 - The pattern gets scaled by a factor of f / z , but angles and ratios of lengths/areas are preserved



$$(X, Y, Z) \rightarrow \left(\frac{fX}{Z}, \frac{fY}{Z} \right)$$

Horizon: Vanishing line of ground plane

$$N_X x + N_Y y + f N_Z = 0$$

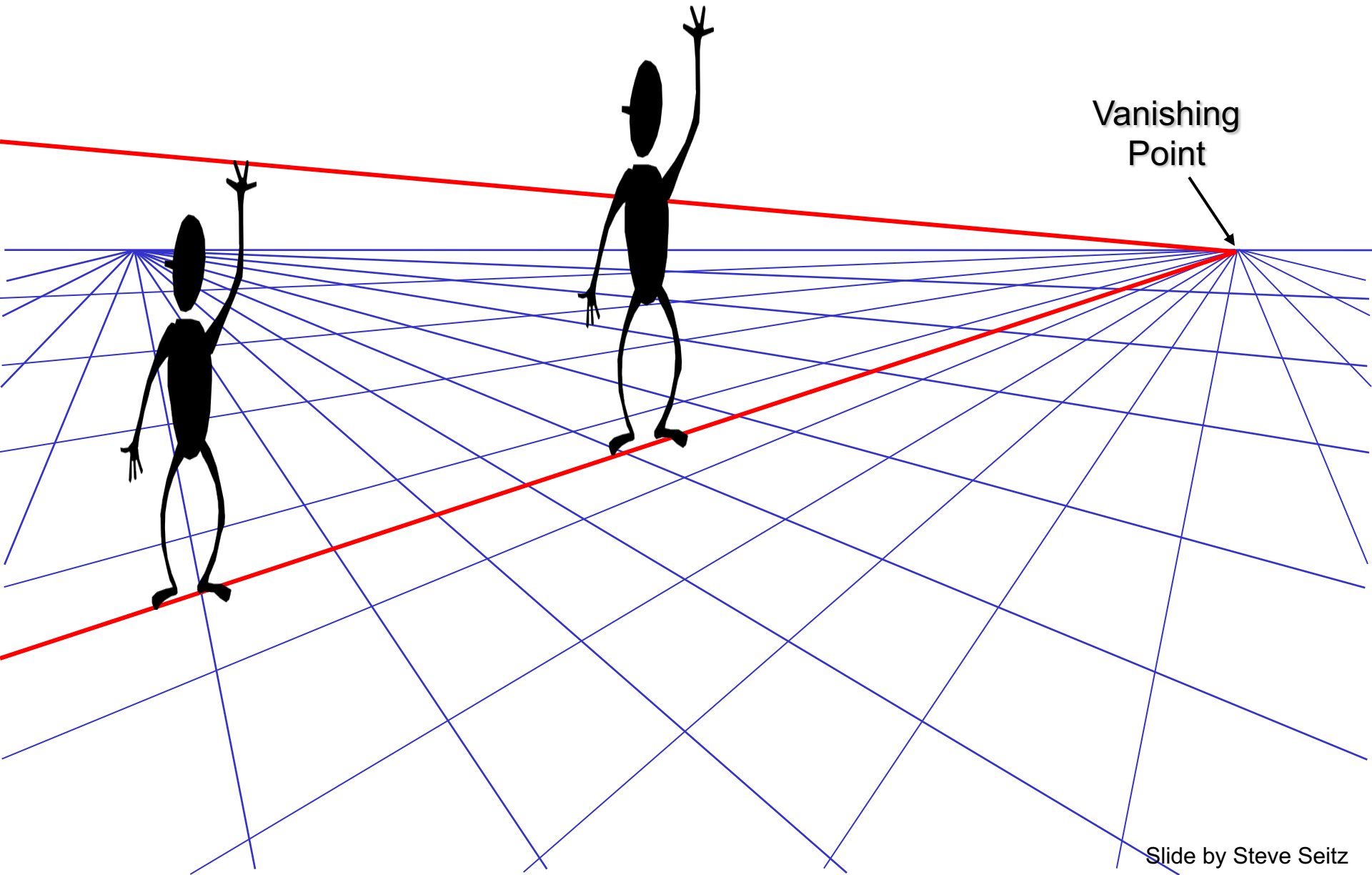


Vanishing lines of planes

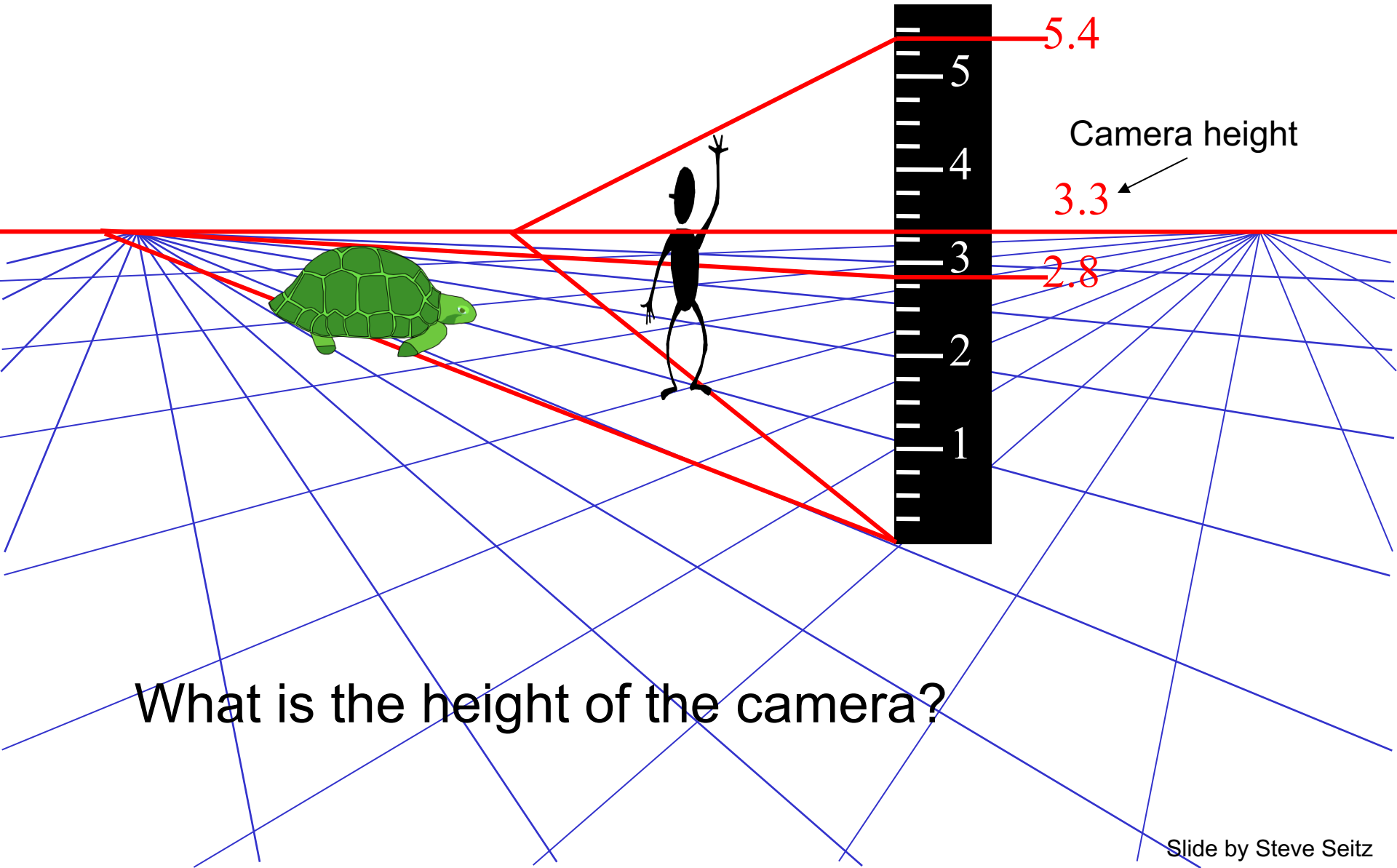


Is the parachutist above or below the camera?

Comparing heights



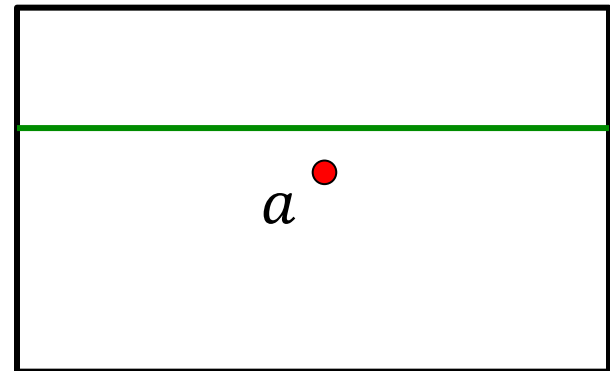
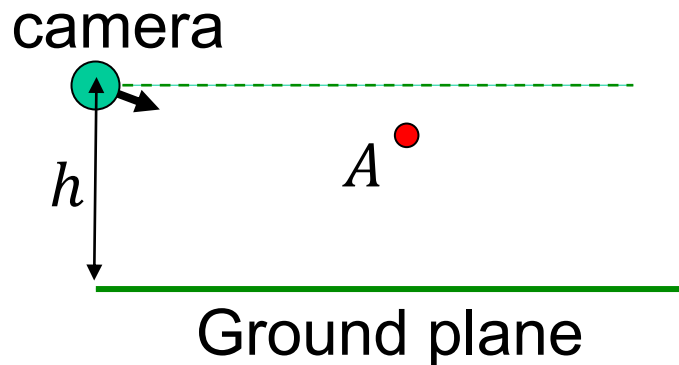
Measuring height



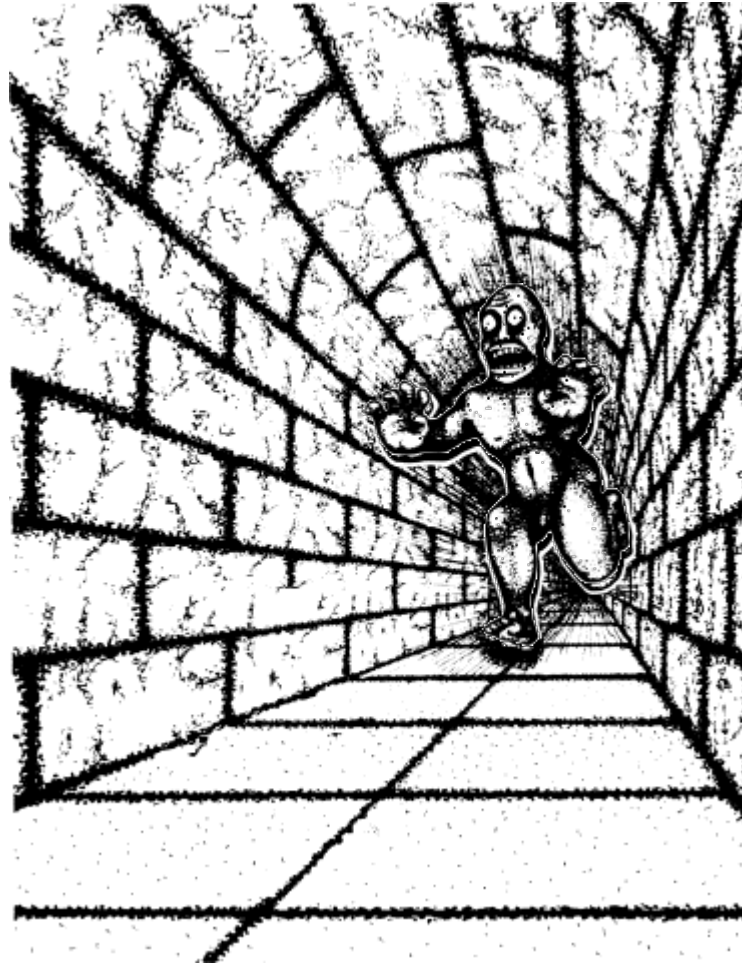
What is the height of the camera?

Horizon: Vanishing line of ground plane

- *Horizon*: vanishing line of the ground plane
 - All points at the same height as the camera project to the horizon
 - Points higher (resp. lower) than the camera project above (resp. below) the horizon
 - Provides way of comparing height of objects

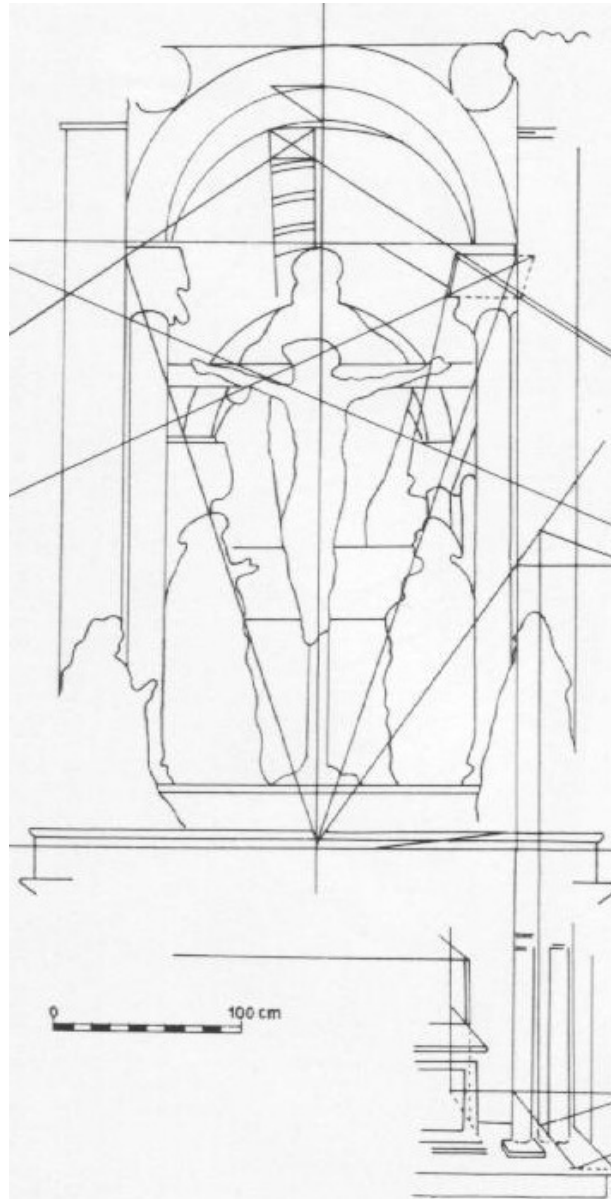
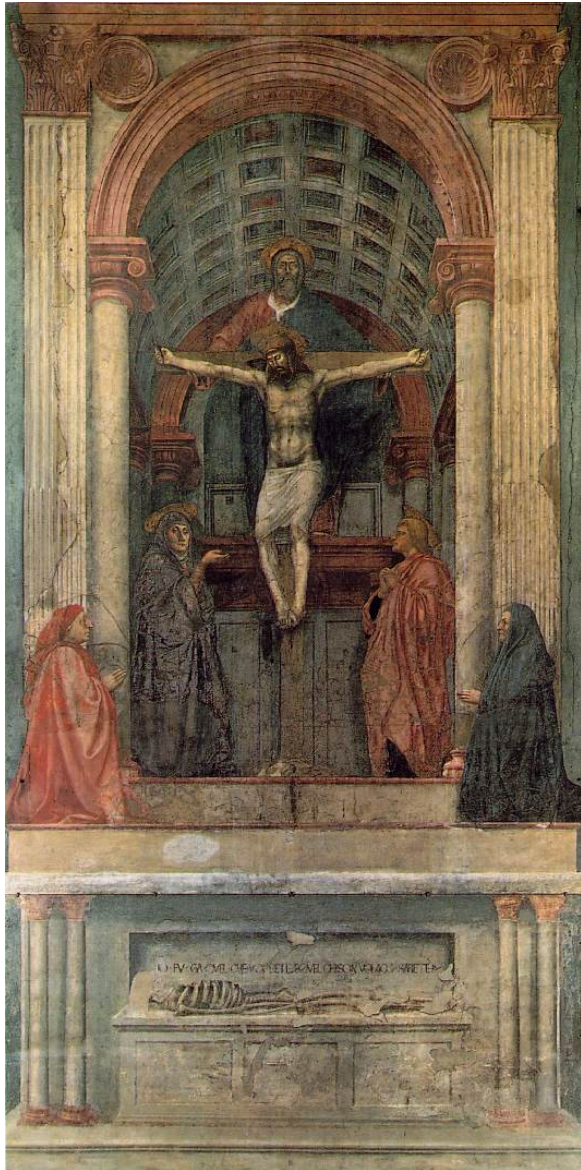


Fun with Projective Geometry



Illusion Credit: RN Shepard, Mind Sights: Original Visual Illusions, Ambiguities, and other Anomalies

Perspective cues in art



Masaccio, *Trinity*, Santa Maria Novella, Florence, 1425-28

One of the first consistent uses of perspective in Western art

Perspective distortion

- What is the shape of the projection of a sphere?

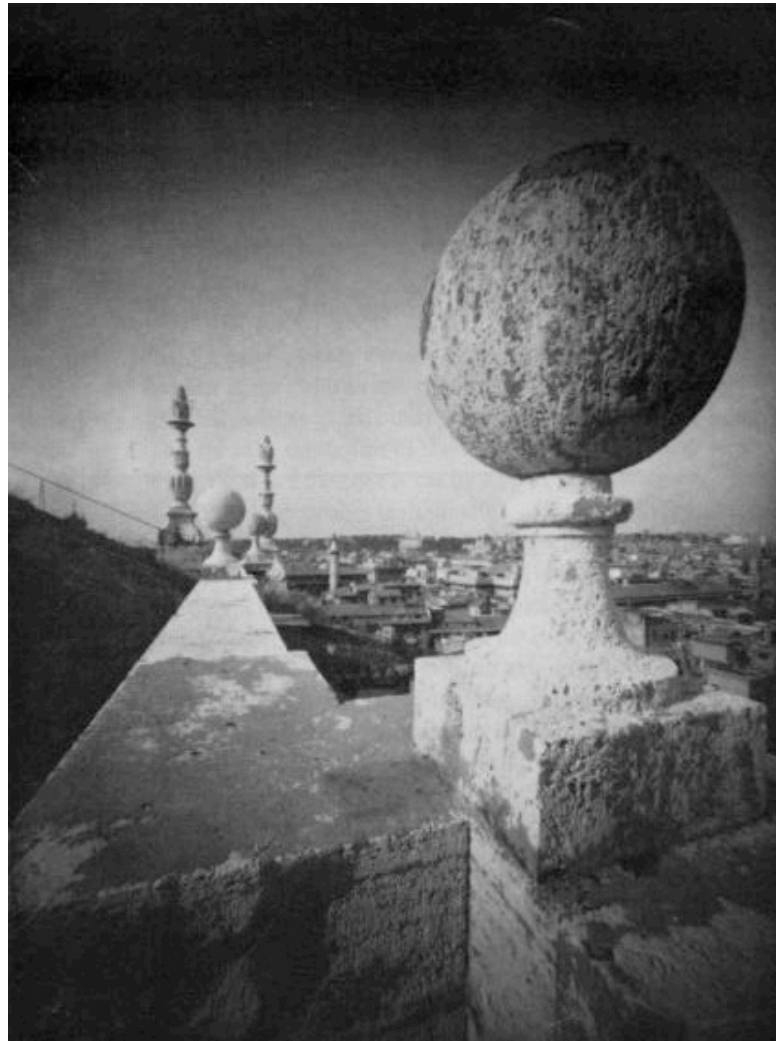
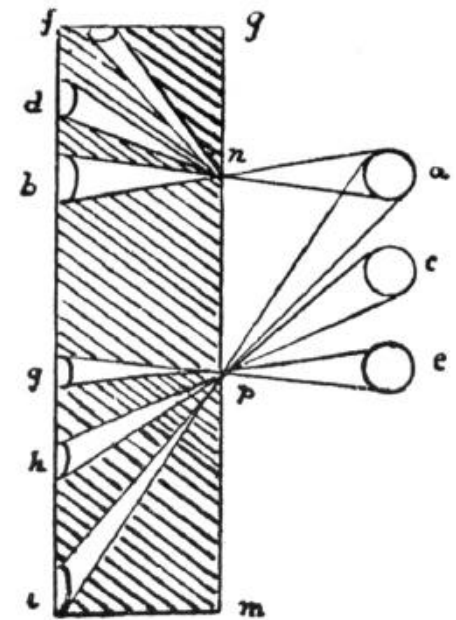
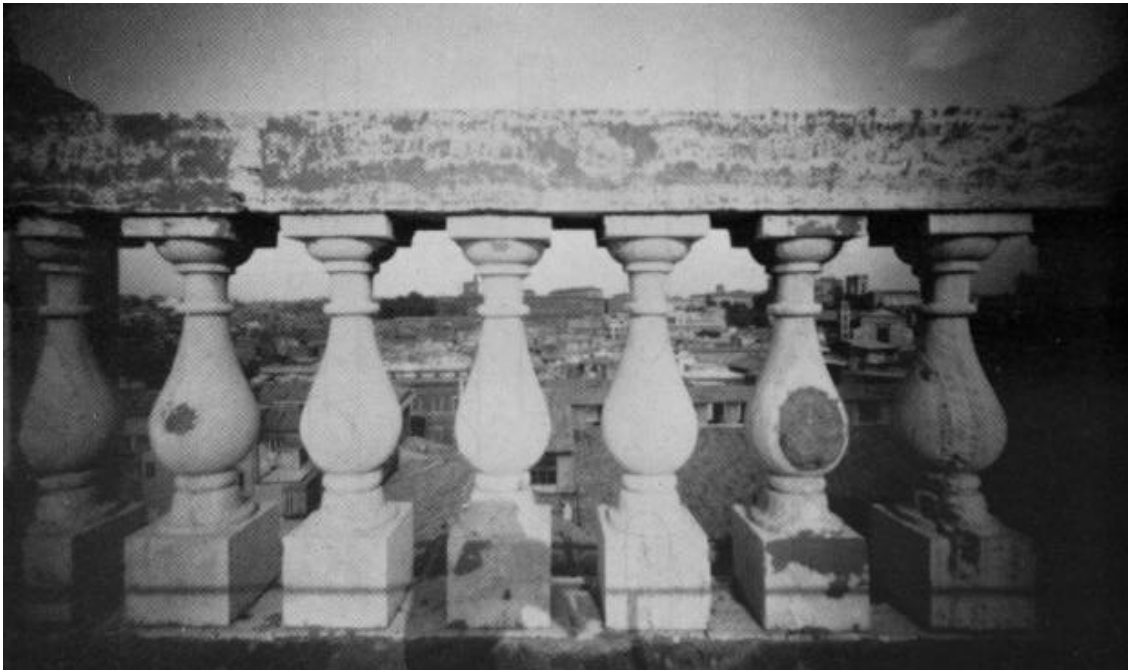


Image source: F. Durand

Perspective distortion

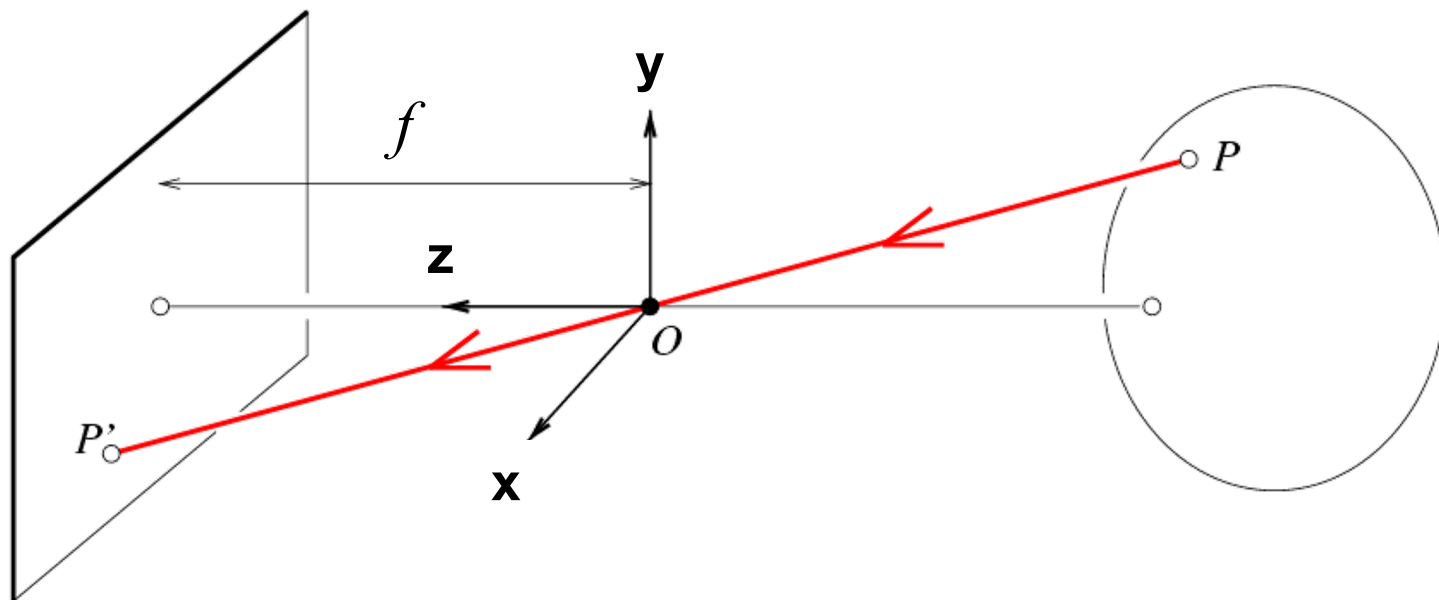
- Are the widths of the projected columns equal?
 - The exterior columns are wider
 - This is not an optical illusion, and is not due to lens flaws
 - Phenomenon pointed out by Da Vinci



Perspective distortion: People



Modeling projection



Projection equation: $(X, Y, Z) \rightarrow \left(\frac{fX}{Z}, \frac{fY}{Z} \right) = (x, y)$

Note: instead of dealing with an image that is upside down, most of the time we will pretend that the image plane is in front of the camera center.

Homogeneous coordinates

$$(X, Y, Z) \rightarrow \left(\frac{fX}{Z}, \frac{fY}{Z} \right)$$

Is this a linear transformation?

- no—division by z is nonlinear

Trick: add one more coordinate:

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

homogeneous image
coordinates

$$(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

homogeneous scene
coordinates

Converting *from* homogeneous coordinates

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w) \qquad \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

Perspective Projection Matrix

Projection is a matrix multiplication using homogeneous coordinates

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \left(f \frac{X}{Z}, f \frac{Y}{Z} \right)$$

divide by the third coordinate

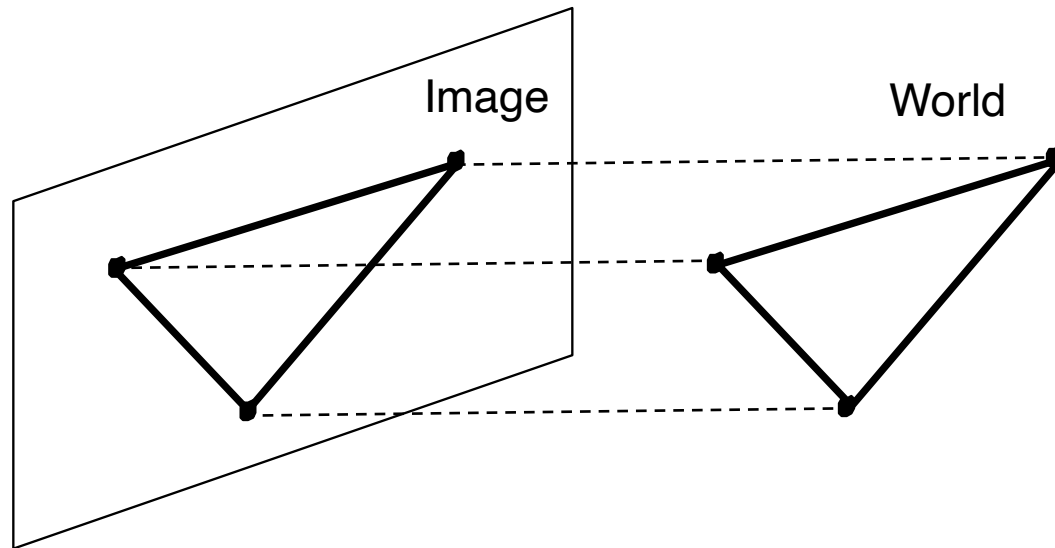
In practice: lots of coordinate transformations...

$$\begin{pmatrix} \text{2D} \\ \text{point} \\ (3 \times 1) \end{pmatrix} = \begin{pmatrix} \text{Camera to} \\ \text{pixel coord.} \\ \text{trans. matrix} \\ (3 \times 3) \end{pmatrix} \begin{pmatrix} \text{Perspective} \\ \text{projection matrix} \\ (3 \times 4) \end{pmatrix} \begin{pmatrix} \text{World to} \\ \text{camera coord.} \\ \text{trans. matrix} \\ (4 \times 4) \end{pmatrix} \begin{pmatrix} \text{3D} \\ \text{point} \\ (4 \times 1) \end{pmatrix}$$

Orthographic Projection

Special case of perspective projection

- Distance from center of projection to image plane is infinite
- Also called “parallel projection”



Orthographic Projection

Special case of perspective projection

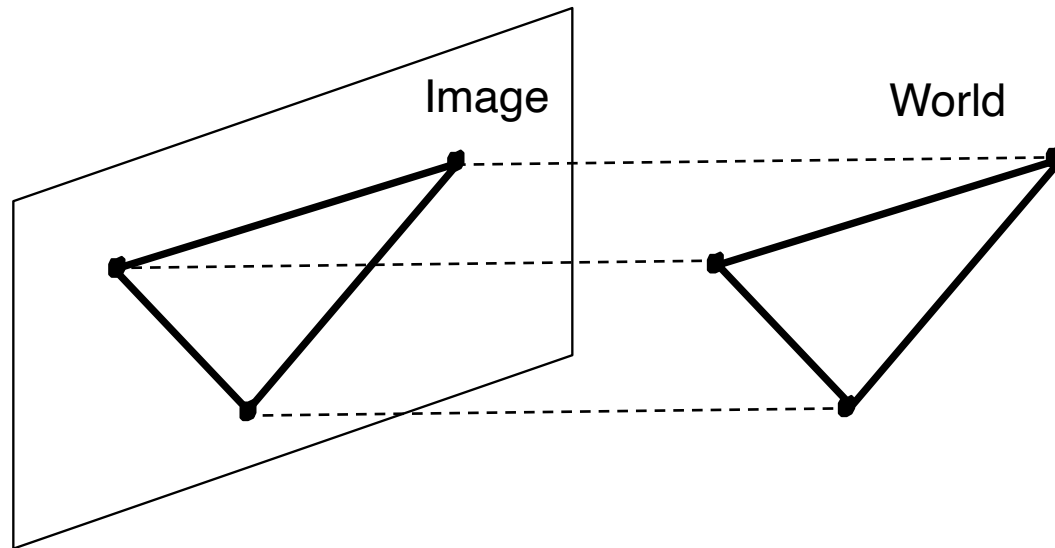
- Distance from center of projection to image plane is infinite
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Orthographic Projection

Special case of perspective projection

- Distance from center of projection to image plane is infinite
- Also called “parallel projection”



- What's the projection matrix?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow (x, y)$$

Recap

