Multi-view geometry

Slides from L. Lazebnik
Structure from motion

Camera 1: $R_1, t_1$
Camera 2: $R_2, t_2$
Camera 3: $R_3, t_3$

Figure credit: Noah Snavely
• **Structure:** Given *known cameras* and projections of the same 3D point in two or more images, compute the 3D coordinates of that point.
Motion: Given a set of known 3D points seen by a camera, compute the camera parameters.
• **Bootstrapping the process:** Given a set of 2D point correspondences in *two images*, compute the camera parameters
Two-view geometry
Epipolar geometry

- **Baseline** – line connecting the two camera centers
- **Epipolar Plane** – plane containing baseline (1D family)
- **Epipoles**
  - intersections of baseline with image planes
  - projections of the other camera center
  - vanishing points of the motion direction
**Epipolar geometry**

- **Baseline** – line connecting the two camera centers
- **Epipolar Plane** – plane containing baseline (1D family)
- **Epipoles**
  - intersections of baseline with image planes
  - projections of the other camera center
  - vanishing points of the motion direction
- **Epipolar Lines** - intersections of epipolar plane with image planes (always come in corresponding pairs)
Example 1

- Converging cameras
Example 2

- Motion parallel to the image plane
Example 3
Example 3

- Motion is perpendicular to the image plane
- Epipole is the “focus of expansion” and the principal point
Motion perpendicular to image plane

http://vimeo.com/48425421
Epipolar constraint

- If we observe a point $x$ in one image, where can the corresponding point $x'$ be in the other image?
Potential matches for $x$ have to lie on the corresponding epipolar line $l'$. 

Potential matches for $x'$ have to lie on the corresponding epipolar line $l$. 

Epipolar constraint
Epipolar constraint example
Epipolar constraint: Calibrated case
Epipolar constraint: Calibrated case

Intrinsic and extrinsic parameters of the cameras are known, world coordinate system is set to that of the first camera.

Then the projection matrices are given by $K[I \mid 0]$ and $K'[R \mid t]$.

We can multiply the projection matrices (and the image points) by the inverse of the calibration matrices to get normalized image coordinates:

$$x_{norm} = K^{-1}x_{pixel} = [I \ 0]X$$
$$x'_{norm} = K'^{-1}x'_pixel = [R \ t]X$$
Epipolar constraint: Calibrated case

Derivation

[Diagram with labeled points and lines]
Epipolar constraint: Calibrated case

Derivation
Epipolar constraint: Calibrated case

The vectors $Rx$, $t$, and $x'$ are coplanar.
Epipolar constraint: Calibrated case

Simplification

\[
[ I \ 0 ] \begin{pmatrix} x \\ 1 \end{pmatrix}
\]

\[
X = (x,1)^T
\]

Recall:

\[
a \times b = [a_x] b
\]

\[
= \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}
\]

The vectors \( Rx, t, \) and \( x' \) are coplanar.
Epipolar constraint: Calibrated case

The vectors $Rx$, $t$, and $x'$ are coplanar

$[I \ 0][x]_1 = X = (x,1)^T$

$\begin{align*}
x' \cdot [t \times (Rx)] &= 0 \\
x'^T[t \times]Rx &= 0 \\
x'^TEx &= 0
\end{align*}$

Essential Matrix
(Longuet-Higgins, 1981)
Epipolar constraint: Calibrated case

- $E \mathbf{x}$ is the epipolar line associated with $\mathbf{x}$ ($l' = E \mathbf{x}$)
  - Recall: a line is given by $ax + by + c = 0$ or
  $$l^T \mathbf{x} = 0,$$
  where $l = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
Epipolar constraint: Calibrated case

- $E \mathbf{x}$ is the epipolar line associated with $\mathbf{x}$ ($l' = E \mathbf{x}$)
- $E^T \mathbf{x}'$ is the epipolar line associated with $\mathbf{x}'$ ($l = E^T \mathbf{x}'$)
- $E \mathbf{e} = 0$ and $E^T \mathbf{e}' = 0$
- $E$ is singular (rank two)
- $E$ has five degrees of freedom

$x'^T E x = 0$
Epipolar constraint: Uncalibrated case

- The calibration matrices $K$ and $K'$ of the two cameras are unknown.
- We can write the epipolar constraint in terms of unknown normalized coordinates:

$$\hat{x}'^T E \hat{x} = 0 \quad \hat{x} = K^{-1} x, \hat{x}' = K'^{-1} x'$$
Epipolar constraint: Uncalibrated case

\[ \hat{x}'^T E \hat{x} = 0 \quad \Rightarrow \quad x'^T F x = 0 \quad \text{with} \quad F = K'^{-T} E K^{-1} \]

\[ \hat{x} = K^{-1} x \]
\[ \hat{x}' = K'^{-1} x' \]

**Fundamental Matrix**
(Faugeras and Luong, 1992)
Epipolar constraint: Uncalibrated case

\( \hat{x}'^T E \hat{x} = 0 \quad \Rightarrow \quad x'^T Fx = 0 \) with \( F = K'^{-T} E K^{-1} \)

- \( Fx \) is the epipolar line associated with \( x \) \((l' = Fx)\)
- \( F^T x' \) is the epipolar line associated with \( x' \) \((l = F^T x')\)
- \( Fe = 0 \) and \( F^T e' = 0 \)
- \( F \) is singular (rank two)
- \( F \) has seven degrees of freedom
Estimating the fundamental matrix
The eight-point algorithm

\[ \mathbf{x} = (u, v, 1)^T, \quad \mathbf{x}' = (u', v', 1) \]
The eight-point algorithm

\[ x = (u, v, 1)^T, \quad x' = (u', v', 1) \]

\[
\begin{bmatrix}
    f_{11} & f_{12} & f_{13} \\
    f_{21} & f_{22} & f_{23} \\
    f_{31} & f_{32} & f_{33}
\end{bmatrix}
\begin{bmatrix}
    u \\
    v \\
    1
\end{bmatrix} = 0
\]

Solve homogeneous linear system using eight or more matches

Enforce rank-2 constraint (take SVD of \( F \) and throw out the smallest singular value)
Problem with eight-point algorithm

\[
\begin{bmatrix}
u'v & u'v' & u'v' & v'v & v' & u & v
\end{bmatrix}
\begin{bmatrix}
f_{11} \\
f_{12} \\
f_{13} \\
f_{21} \\
f_{22} \\
f_{23} \\
f_{31} \\
f_{32}
\end{bmatrix} = -1
\]
Problem with eight-point algorithm

Poor numerical conditioning
Can be fixed by rescaling the data

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\[
\begin{bmatrix}
    f_{11} \\
    f_{12} \\
    f_{13} \\
    f_{21} \\
    f_{22} \\
    f_{23} \\
    f_{31} \\
    f_{32}
\end{bmatrix} = -1
\]
The normalized eight-point algorithm

(Hartley, 1995)

- Center the image data at the origin, and scale it so the mean squared distance between the origin and the data points is 2 pixels
- Use the eight-point algorithm to compute $F$ from the normalized points
- Enforce the rank-2 constraint (for example, take SVD of $F$ and throw out the smallest singular value)
- Transform fundamental matrix back to original units: if $T$ and $T'$ are the normalizing transformations in the two images, than the fundamental matrix in original coordinates is $T'^T F T$
Seven-point algorithm

- Set up least squares system with seven pairs of correspondences and solve for null space (two vectors) using SVD
- Solve for linear combination of null space vectors that satisfies $\det(F)=0$

Source: D. Hoiem
Nonlinear estimation

- Linear estimation minimizes the sum of squared *algebraic* distances between points $x'_i$ and epipolar lines $Fx_i$ (or points $x_i$ and epipolar lines $F^T x'_i$):
  \[ \sum_{i=1}^{N} \left( x'_i^T F x_i \right)^2 \]

- Nonlinear approach: minimize sum of squared *geometric* distances
  \[ \sum_{i=1}^{N} \left[ d^2 (x'_i, Fx_i) + d^2 (x_i, F^T x'_i) \right] \]
Comparison of estimation algorithms

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The Fundamental Matrix Song

http://danielwedge.com/fmatrix/
From epipolar geometry to camera calibration

- Estimating the fundamental matrix is known as “weak calibration”
- If we know the calibration matrices of the two cameras, we can estimate the essential matrix: $E = K'^TFK$
- The essential matrix gives us the relative rotation and translation between the cameras, or their extrinsic parameters
- Alternatively, if the calibration matrices are known, the five-point algorithm can be used to estimate relative camera pose
Recap (Two-view Geometry)

**Epipolar geometry terminology**

\[ x' \begin{bmatrix} t_x \end{bmatrix} R x = 0 \]

**Essential Matrix**

\[ x' F x = 0 \text{ with } F = K'^{-T} E K^{-1} \]

**Fundamental Matrix**

Properties of Essential and Fundamental Matrix

Estimation of Fundamental Matrix from point correspondences
Questions?

\[ E = [t \times]R \]

- Why does \( E \) only have 5 degree of freedom?
- Why is \( E e = 0 \)? Or why is \( E^T e' = 0 \)?
- Why does \( E \) have rank 2?
- What are the singular values of \( E \)?
- Can you recover \( t \) and \( R \) from \( E \)?
Translation and Rotation from $E$

**Result 9.18.** Suppose that the SVD of $E$ is $U \text{diag}(1, 1, 0)V^T$. Using the notation of (9.13), there are (ignoring signs) two possible factorizations $E = SR$ as follows:

$$S = UZU^T \quad R = UWV^T \quad \text{or} \quad UW^TV^T.$$  \hspace{1cm} (9.14)

**Proof.** That the given factorization is valid is true by inspection. That there are no other factorizations is shown as follows. Suppose $E = SR$. The form of $S$ is determined by the fact that its left null-space is the same as that of $E$. Hence $S = UZU^T$. The rotation $R$ may be written as $UXV^T$, where $X$ is some rotation matrix. Then

$$U \text{diag}(1, 1, 0)V^T = E = SR = (UZU^T)(UXV^T) = U(ZX)V^T$$

from which one deduces that $ZX = \text{diag}(1, 1, 0)$. Since $X$ is a rotation matrix, it follows that $X = W$ or $X = W^T$, as required. \hfill \square

$$W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Source: Hartley and Zisserman