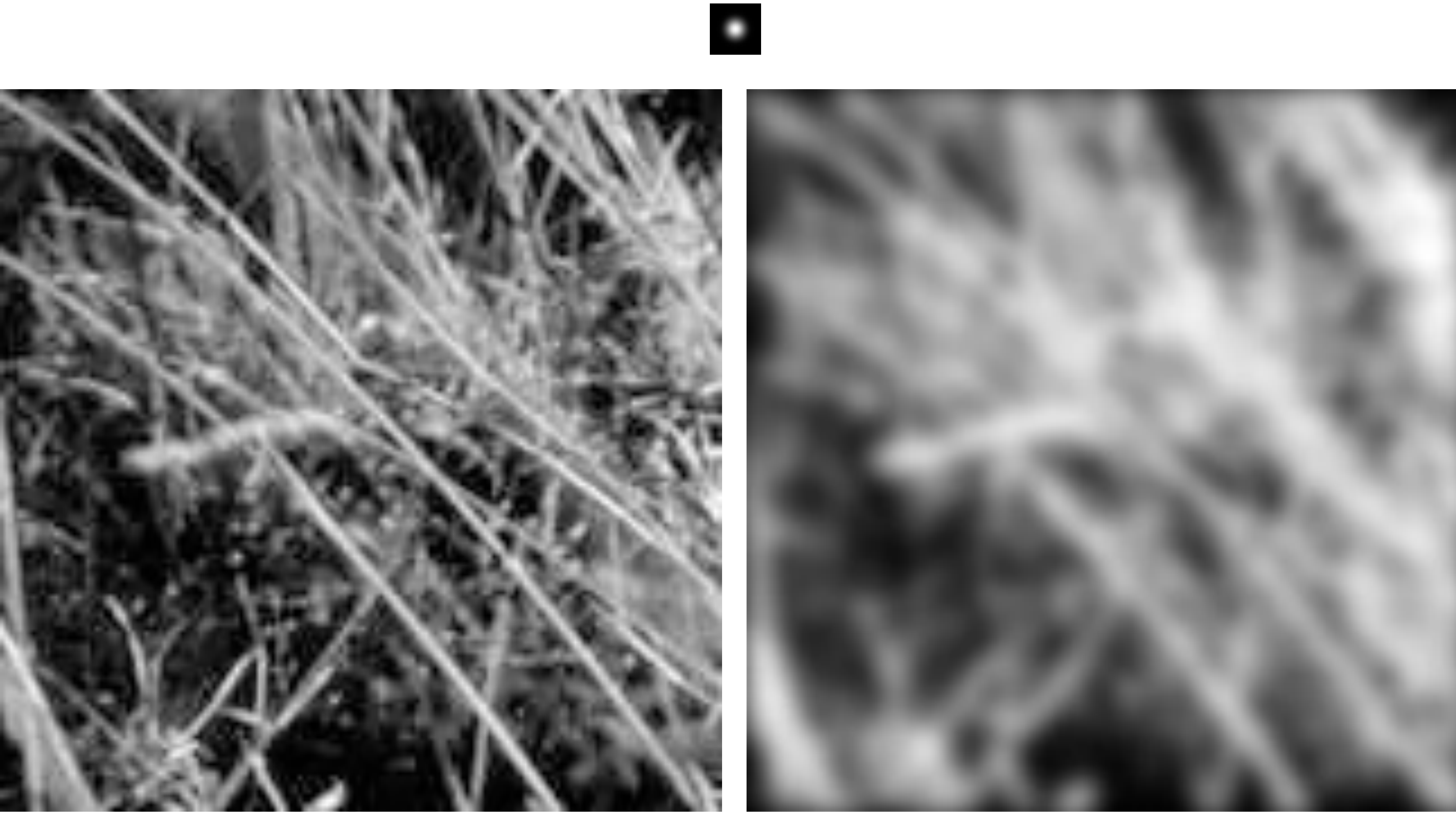
Linear filtering



Slides from Lana Lazebnik

Motivation: Image denoising

- How can we reduce noise in a photograph?

Slide from Lana Lazebnik

Moving average

- Let's replace each pixel with a *weighted average* of its neighborhood
- The weights are called the *filter kernel*
- What are the weights for the average of a 3x3 neighborhood?

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

"box filter"

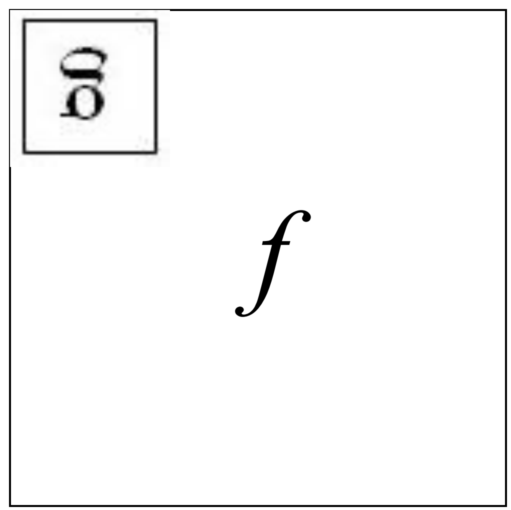
Source: D. Lowe

Defining convolution

- Let *f* be the image and *g* be the kernel. The output of convolving *f* with *g* is denoted *f * g*.

$$(f * g)[m,n] = \sum_{k,l} f[m-k, n-l]\, g[k,l]$$

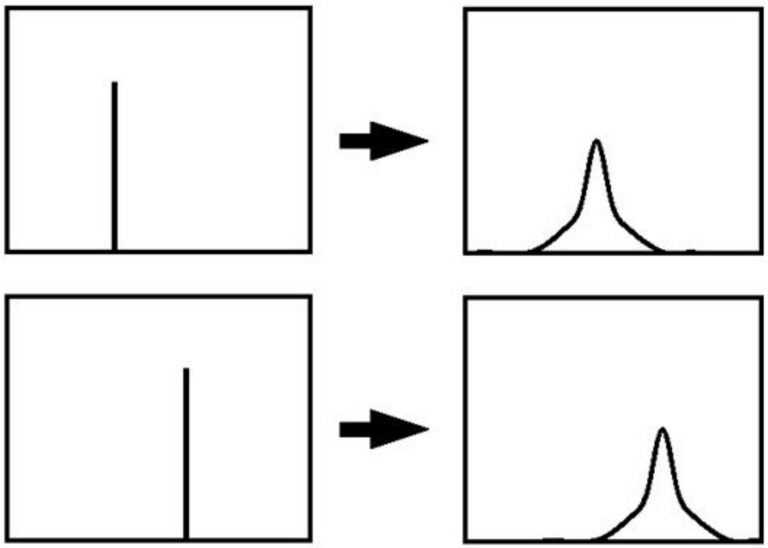
Convention: kernel is “flipped”



Source: F. Durand

Key properties

- **Shift invariance:** same behavior regardless of pixel location: filter(shift(f)) = shift(filter(f))



- **Linearity:** filter($f_1 + f_2$) = filter(f_1) + filter(f_2)

- Theoretical result: any linear shift-invariant operator can be represented as a convolution

Slide from Lana Lazebnik

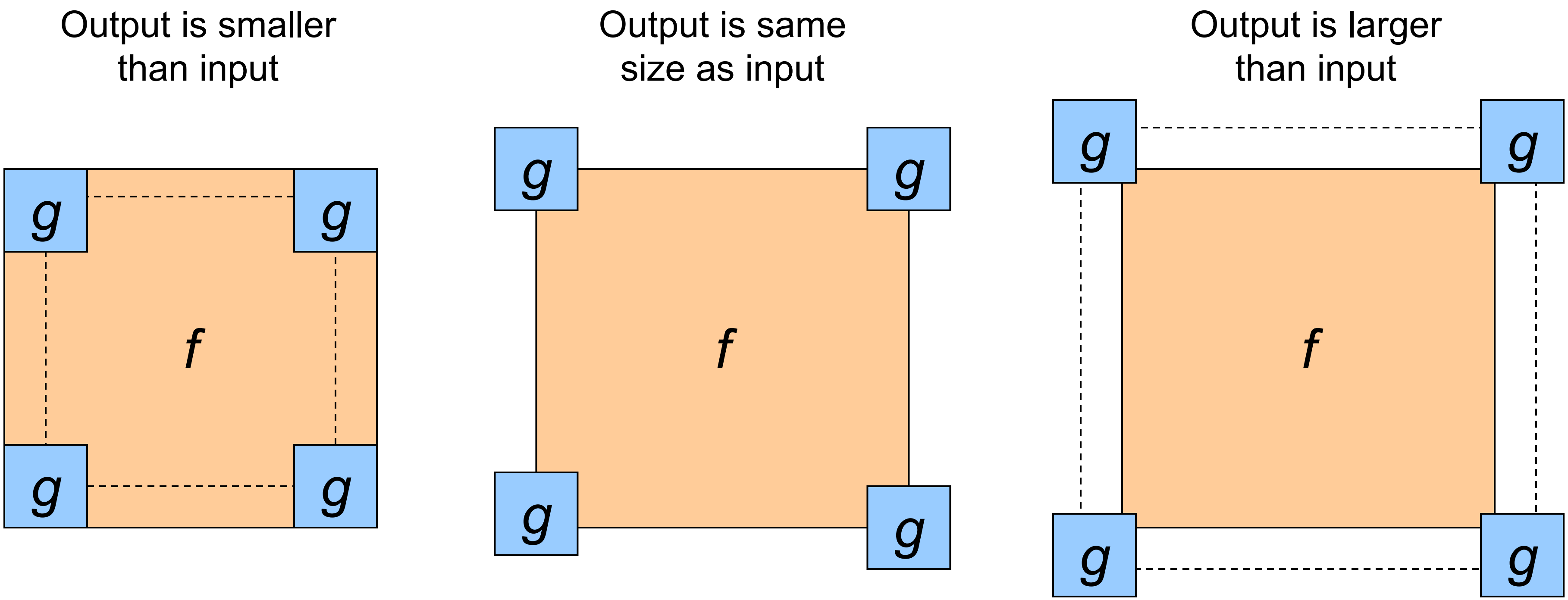
Properties in more detail

- Commutative: $a * b = b * a$
  - Conceptually no difference between filter and signal
- Associative: $a * (b * c) = (a * b) * c$
  - Often apply several filters one after another: $(((a * b_1) * b_2) * b_3)$
  - This is equivalent to applying one filter: $a * (b_1 * b_2 * b_3)$
- Distributes over addition: $a * (b + c) = (a * b) + (a * c)$
- Scalars factor out: $ka * b = a * kb = k (a * b)$
- Identity: unit impulse $e = [\ldots, 0, 0, 1, 0, 0, \ldots]$, $a * e = a$

Slide from Lana Lazebnik

Dealing with edges

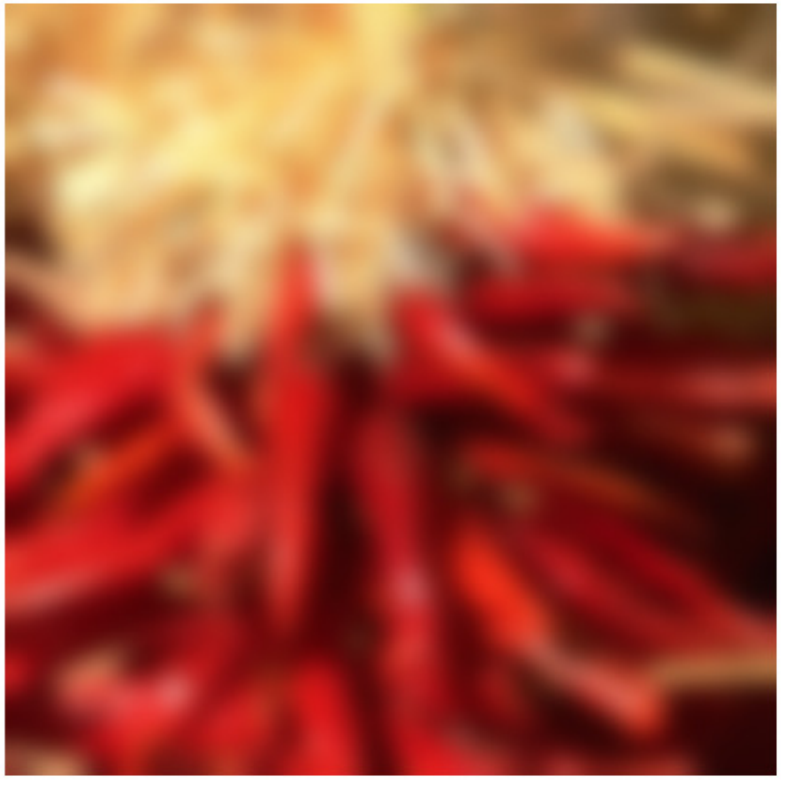
- If we convolve image *f* with filter *g*, what is the size of the output?



Slide from Lana Lazebnik

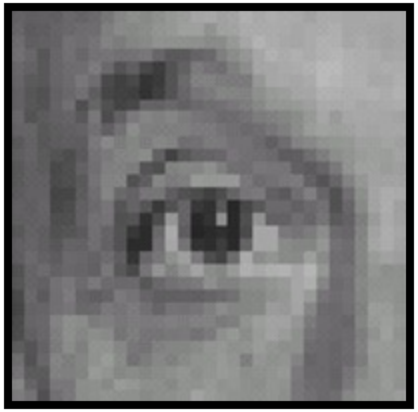
Dealing with edges

- If the filter window falls off the edge of the image, we need to pad the image
  - Zero pad (or clip filter)
  - Wrap around
  - Copy edge
  - Reflect across edge



Source: S. Marschner

Practice with linear filters



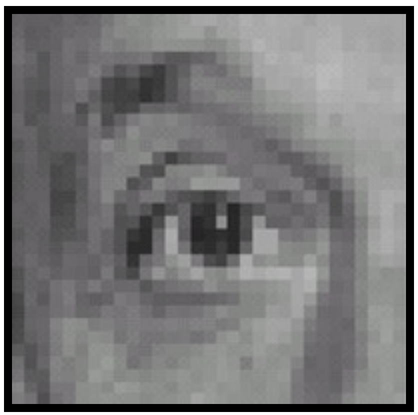
Original

0	0	0
0	1	0
0	0	0

?

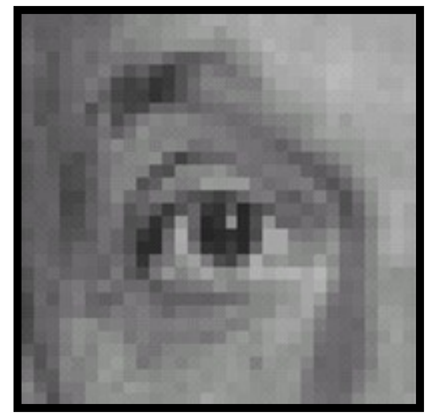
Source: D. Lowe

Practice with linear filters



Original

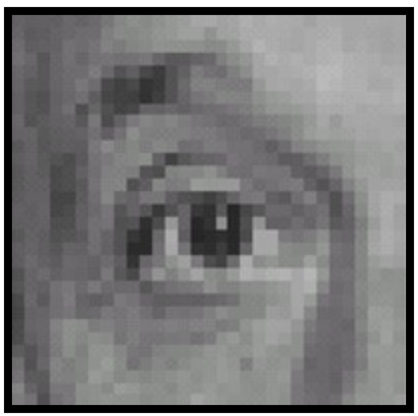
0	0	0
0	1	0
0	0	0



Filtered
(no change)

Source: D. Lowe

Practice with linear filters



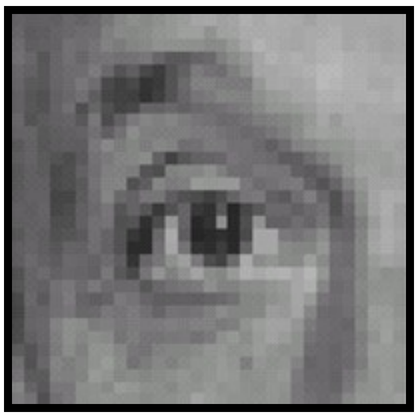
Original

0	0	0
0	0	1
0	0	0

?

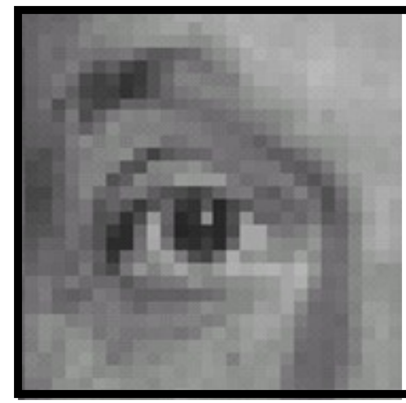
Source: D. Lowe

Practice with linear filters



Original

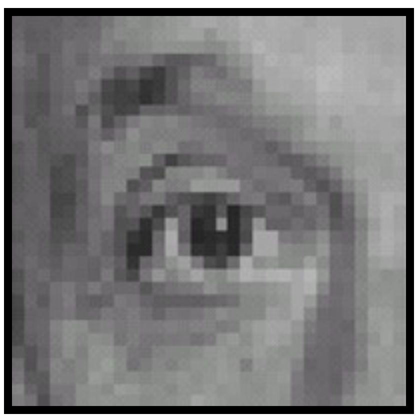
0	0	0
0	0	1
0	0	0



Shifted *left*
By 1 pixel

Source: D. Lowe

Practice with linear filters



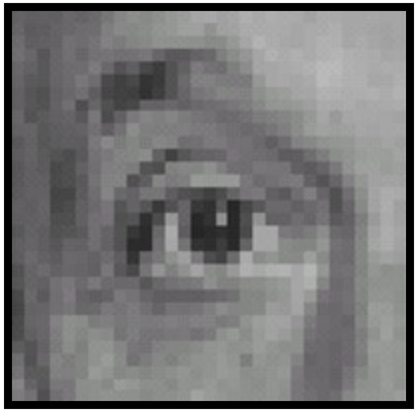
Original

$$\frac{1}{9}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$

?

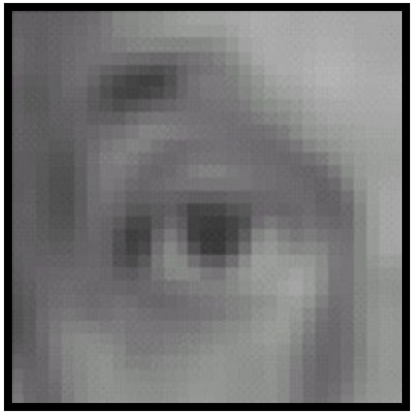
Source: D. Lowe

Practice with linear filters



Original

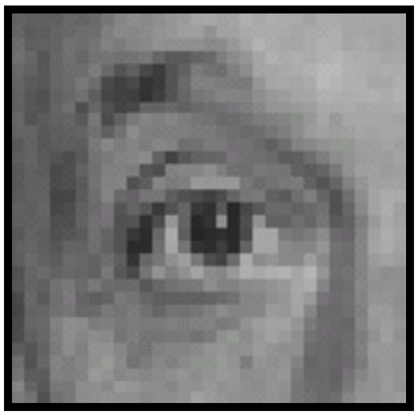
$$\frac{1}{9}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$



Blur (with a box filter)

Source: D. Lowe

Practice with linear filters



Original

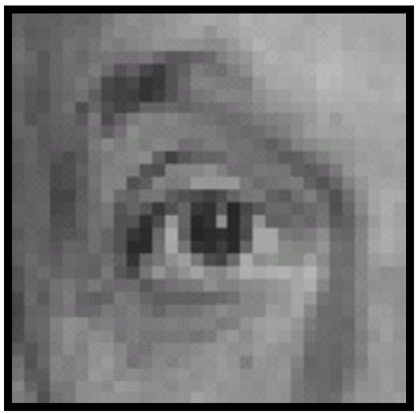
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

?

(Note that filter sums to 1)

Source: D. Lowe

Practice with linear filters



Original

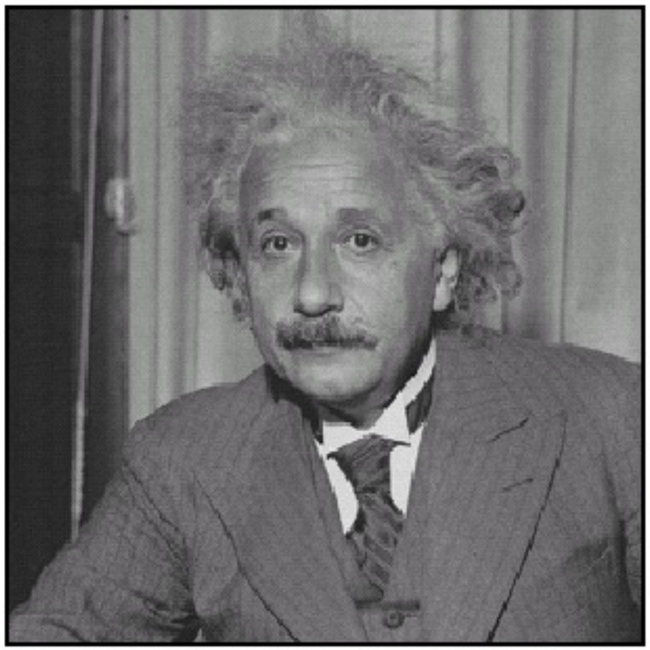
$$\begin{bmatrix}0&0&0\\0&2&0\\0&0&0\end{bmatrix} - \frac{1}{9}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

Sharpening filter

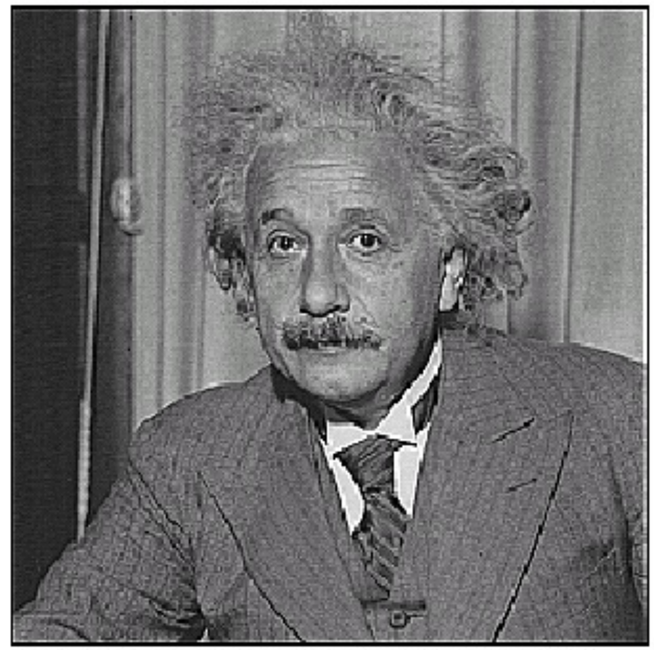
- Accentuates differences with local average

Source: D. Lowe

Sharpening



before



after

Source: D. Lowe

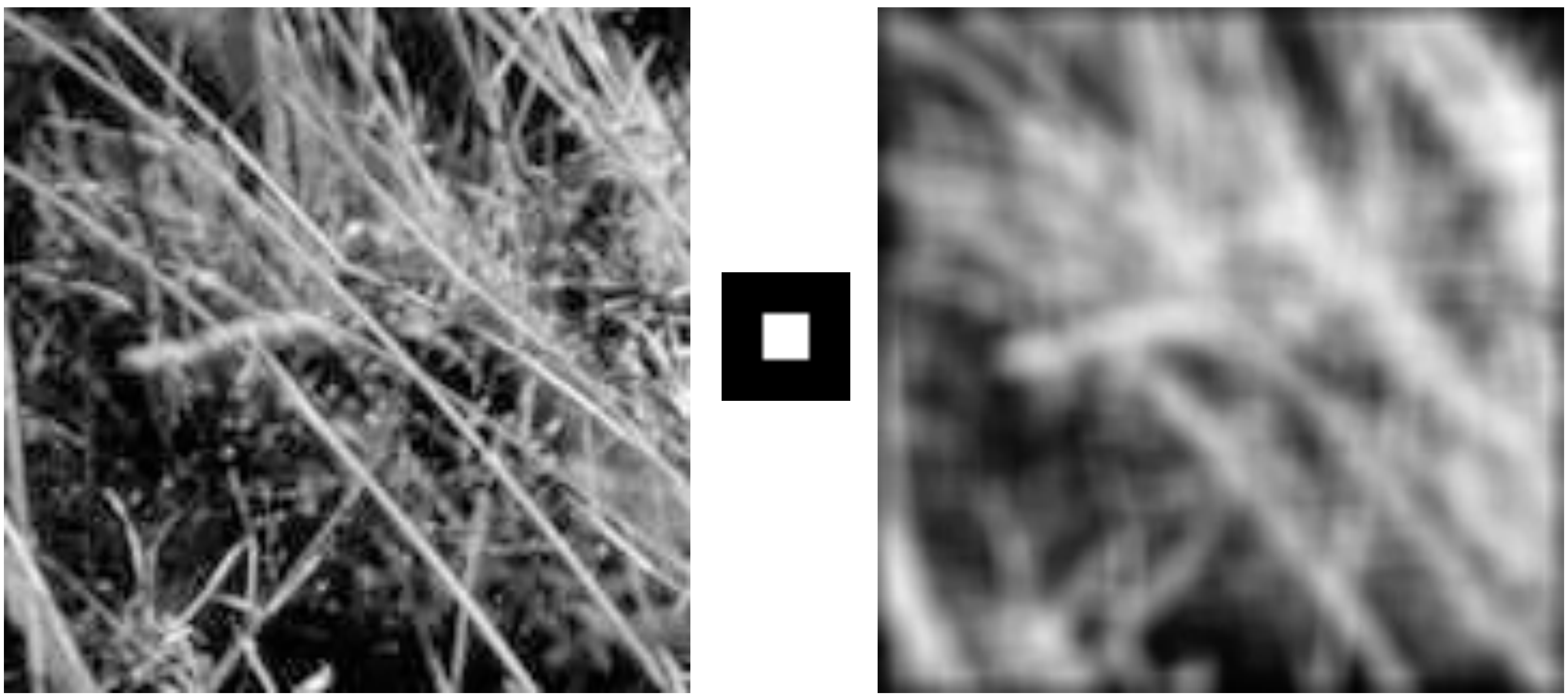
Sharpening

What does blurring take away?

Let's add it back in.

Smoothing with box filter revisited

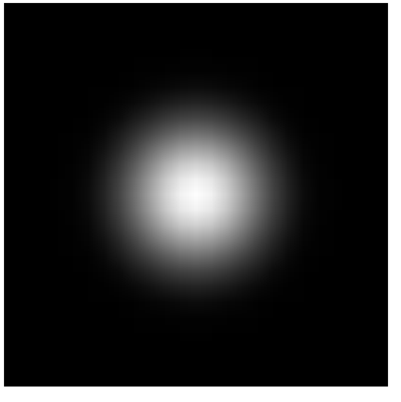
- What's wrong with this picture?
- What's the solution?



Source: D. Forsyth

Smoothing with box filter revisited

- What's wrong with this picture?
- What's the solution?
  - To eliminate edge effects, weight contribution of neighborhood pixels according to their closeness to the center

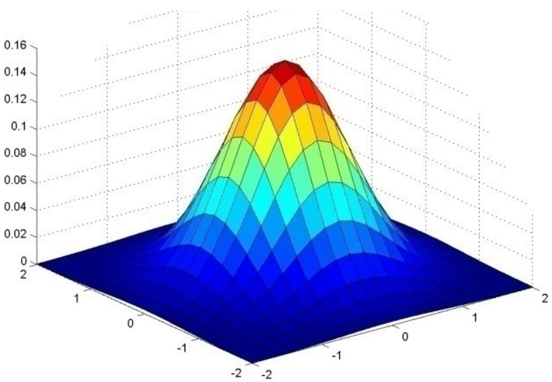


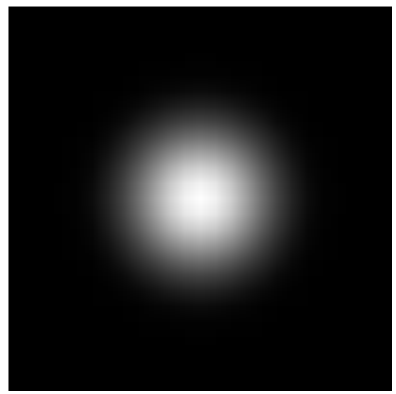
"fuzzy blob"

Slide from Lana Lazebnik

Gaussian Kernel

$$G_\sigma = \frac{1}{2\pi\sigma^2} e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}$$





0.003	0.013	0.022	0.013	0.003
0.013	0.059	0.097	0.059	0.013
0.022	0.097	0.159	0.097	0.022
0.013	0.059	0.097	0.059	0.013
0.003	0.013	0.022	0.013	0.003

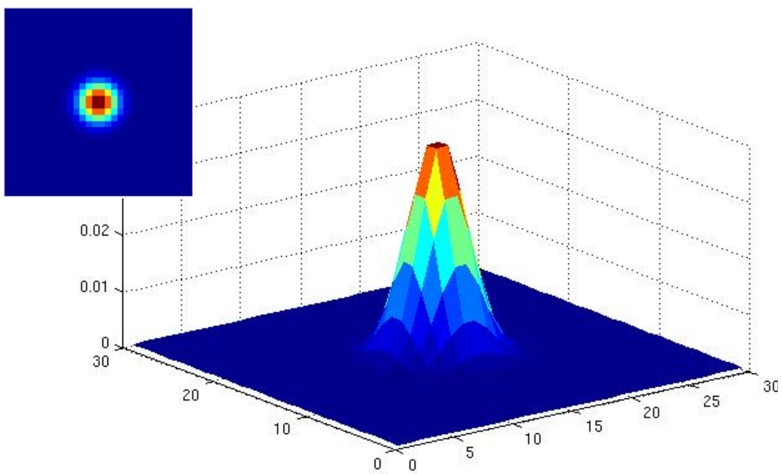
5 x 5, σ = 1

- Constant factor at front makes volume sum to 1 (can be ignored when computing the filter values, as we should renormalize weights to sum to 1 in any case)

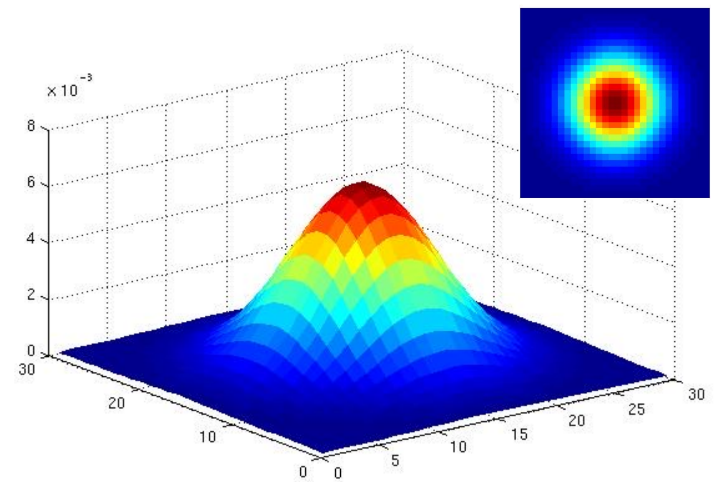
Source: C. Rasmussen

Gaussian Kernel

$$G_\sigma = \frac{1}{2\pi\sigma^2} e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}$$



σ = 2 with 30 x 30 kernel



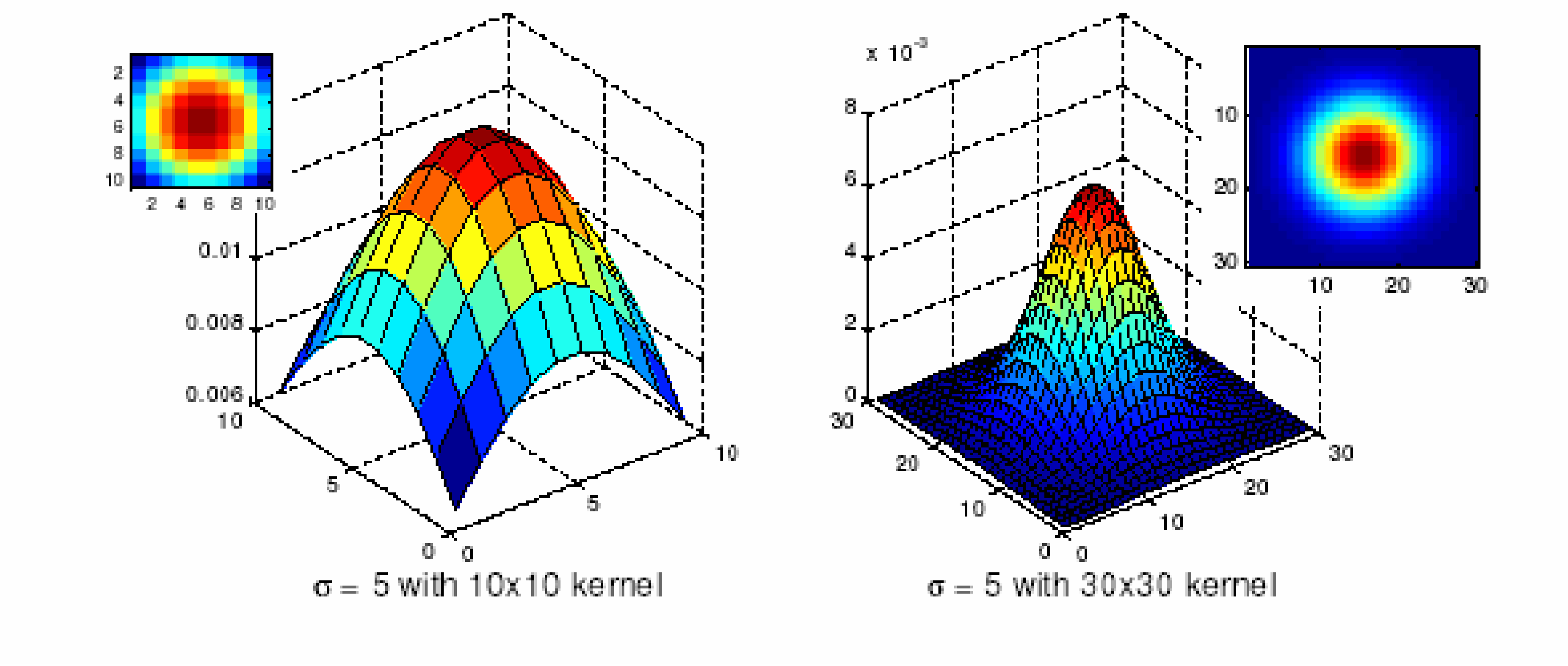
σ = 5 with 30 x 30 kernel

- Standard deviation σ: determines extent of smoothing

Source: K. Grauman

Choosing kernel width

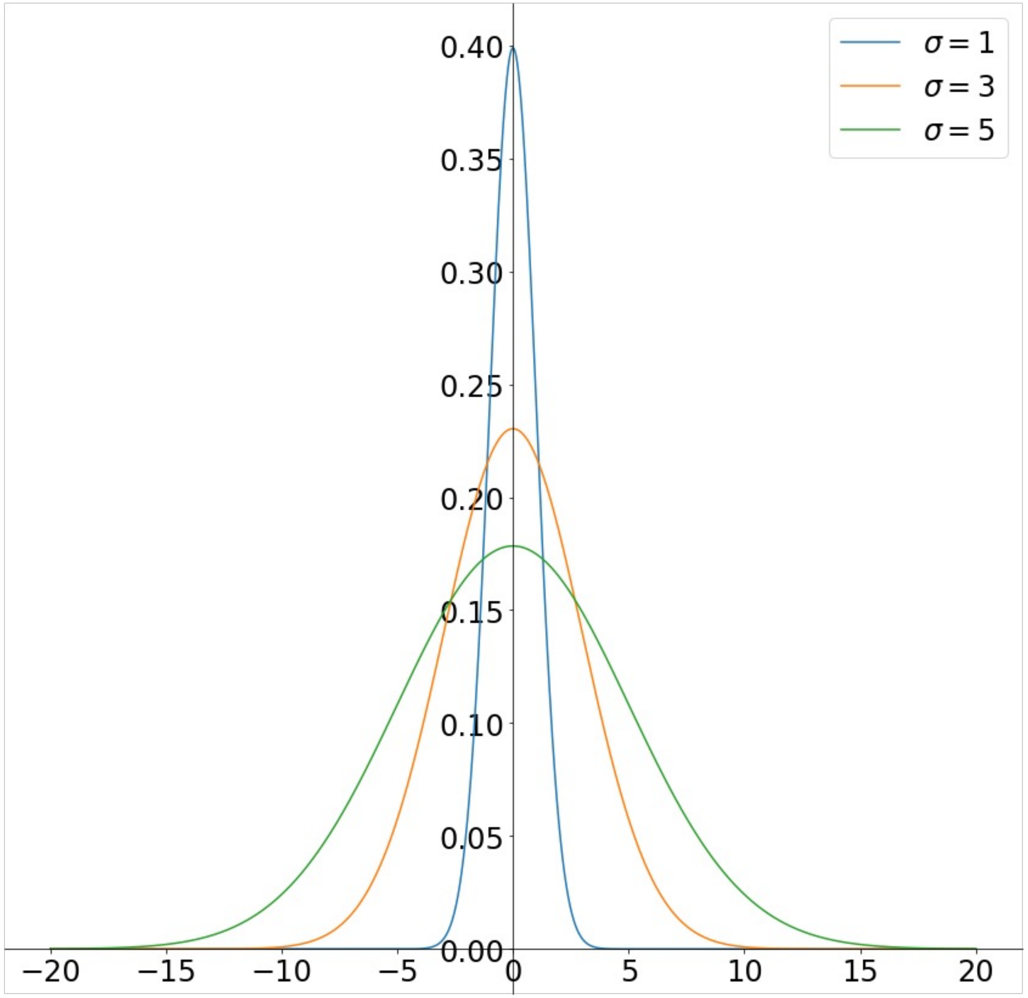
- The Gaussian function has infinite support, but discrete filters use finite kernels



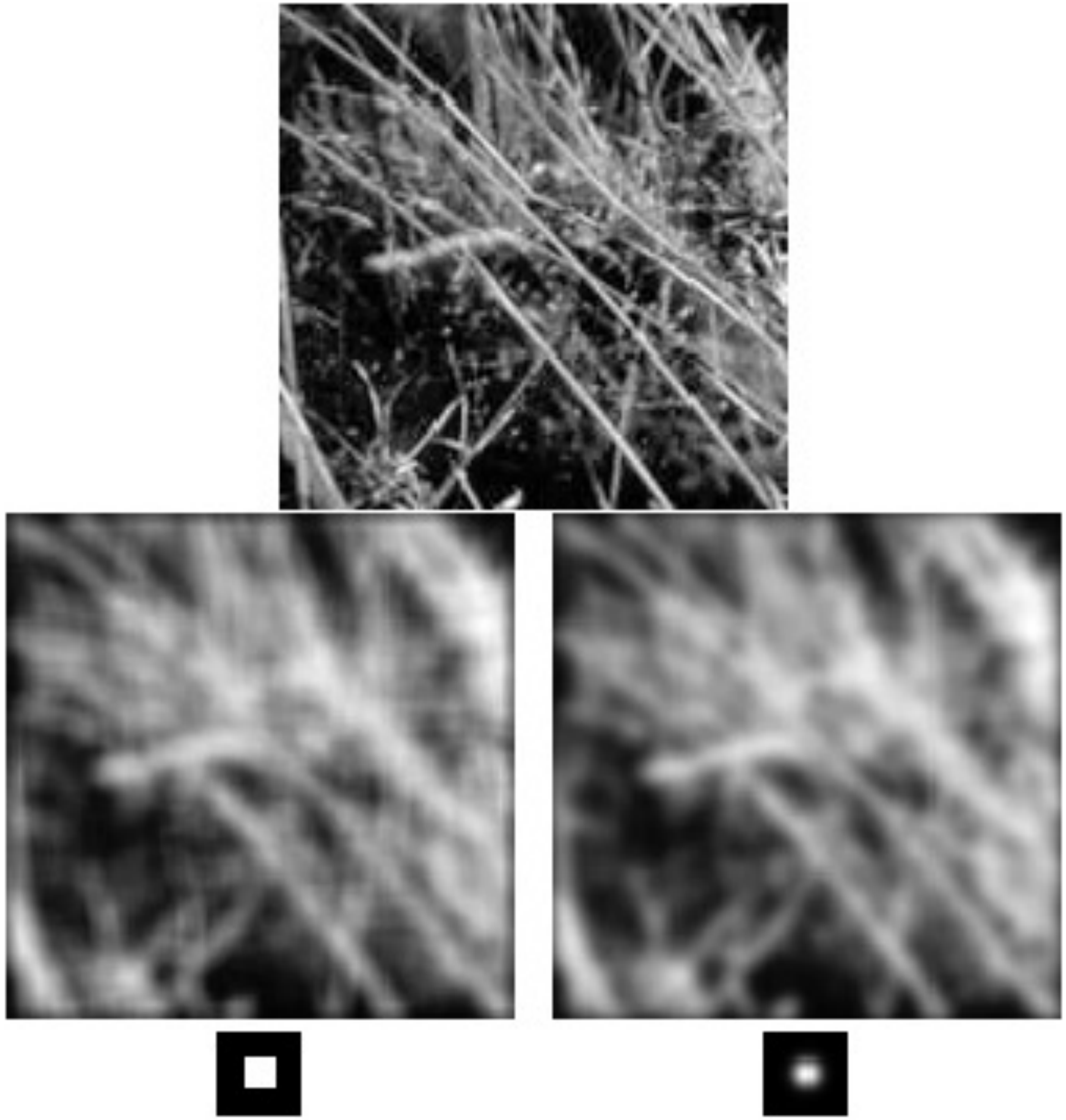
Source: K. Grauman

Choosing kernel width

- Rule of thumb: set filter half-width to about 3σ



Gaussian vs. box filtering



Slide from Lana Lazebnik

Gaussian filters

- Remove high-frequency components from the image (*low-pass filter*)
- Convolution with self is another Gaussian
  - So can smooth with small-σ kernel, repeat, and get same result as larger-σ kernel would have
  - Convolving two times with Gaussian kernel with std. dev. σ is same as convolving once with kernel with std. dev. $\sigma\sqrt{2}$
- *Separable* kernel
  - Factors into product of two 1D Gaussians
  - Discrete example:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

Source: K. Grauman

Separability of the Gaussian filter

$$G_\sigma(x,y) = \frac{1}{2\pi\sigma^2} e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\right) \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}\right)$$

The 2D Gaussian can be expressed as the product of two functions, one a function of x and the other a function of y.

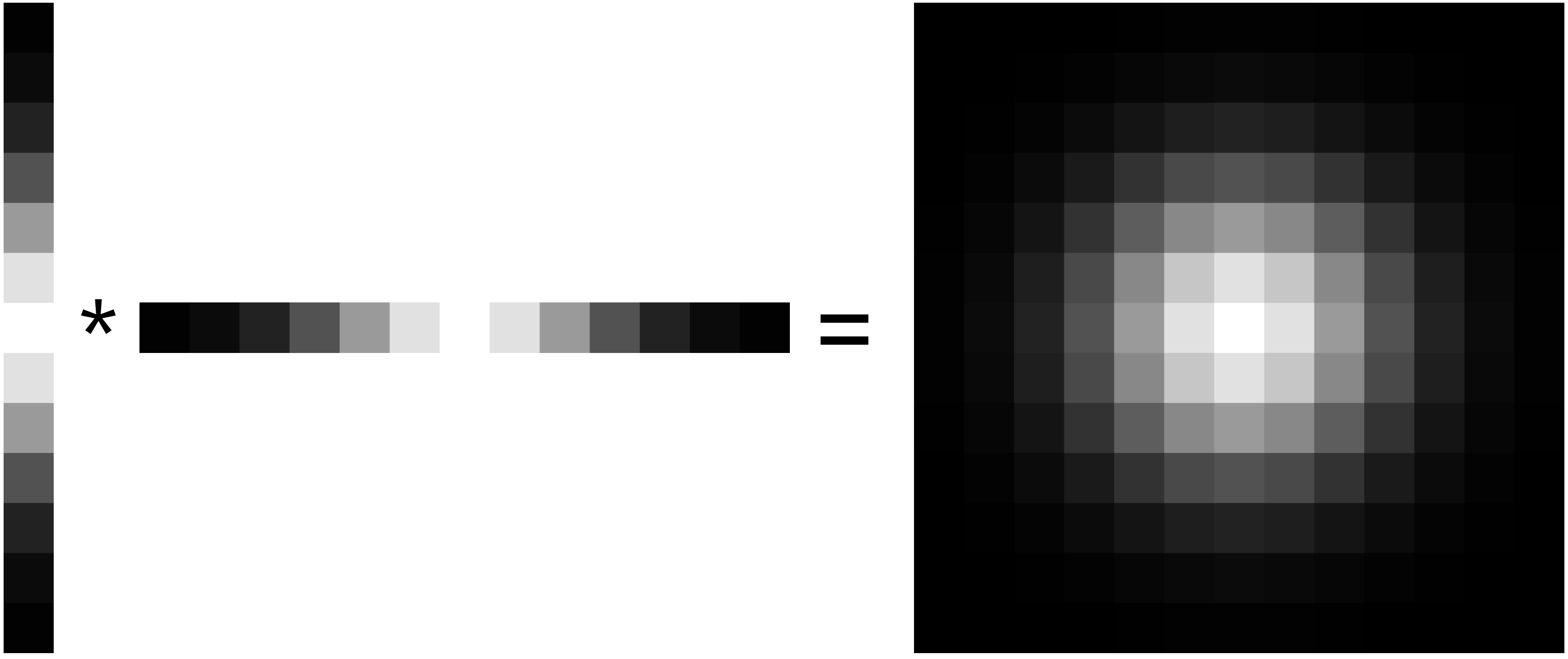
In this case the two functions are the (identical) 1D Gaussian.

Source: D. Lowe

Separability

1D Gaussian $\star$ 1D Gaussian = 2D Gaussian

Image $\star$ 2D Gauss = Image $\star$ (1D Gauss $\star$ 1D Gauss)
= (Image $\star$ 1D Gauss) $\star$ 1D Gauss



Why is separability useful?

- Separability means that a 2D convolution can be reduced to two 1D convolutions (one along rows and one along columns)
- What is the complexity of filtering an $n \times n$ image with an $m \times m$ kernel?
  - $O(n^2 m^2)$
- What if the kernel is separable?
  - $O(n^2 m)$

Slide from Lana Lazebnik

Noise

Original

Salt and pepper noise

Impulse noise

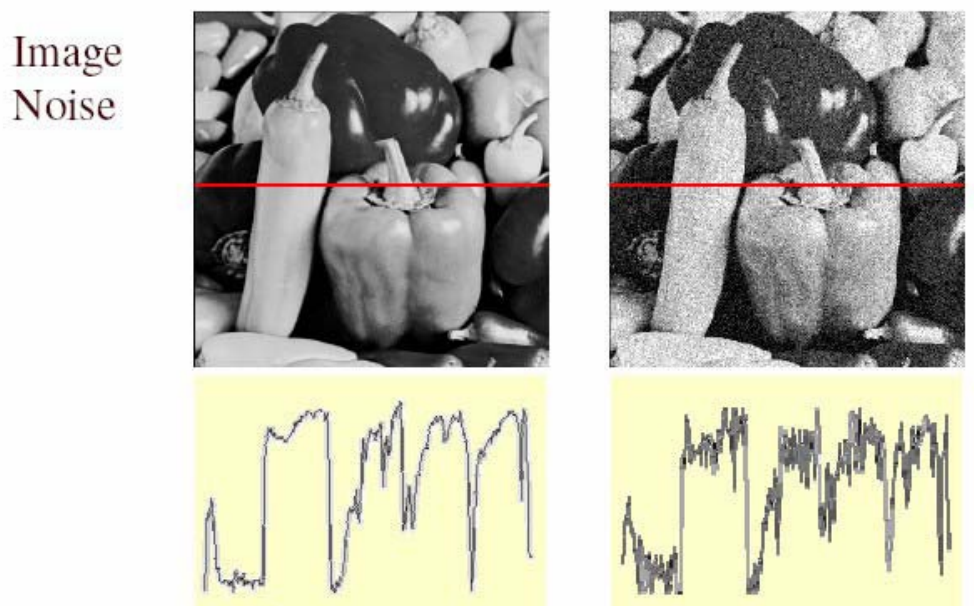
Gaussian noise

- **Salt and pepper noise**: contains random occurrences of black and white pixels
- **Impulse noise:** contains random occurrences of white pixels
- **Gaussian noise**: variations in intensity drawn from a Gaussian normal distribution

Source: S. Seitz

Gaussian noise

- Mathematical model: sum of many independent factors
- Good for small standard deviations
- Assumption: independent, zero-mean noise

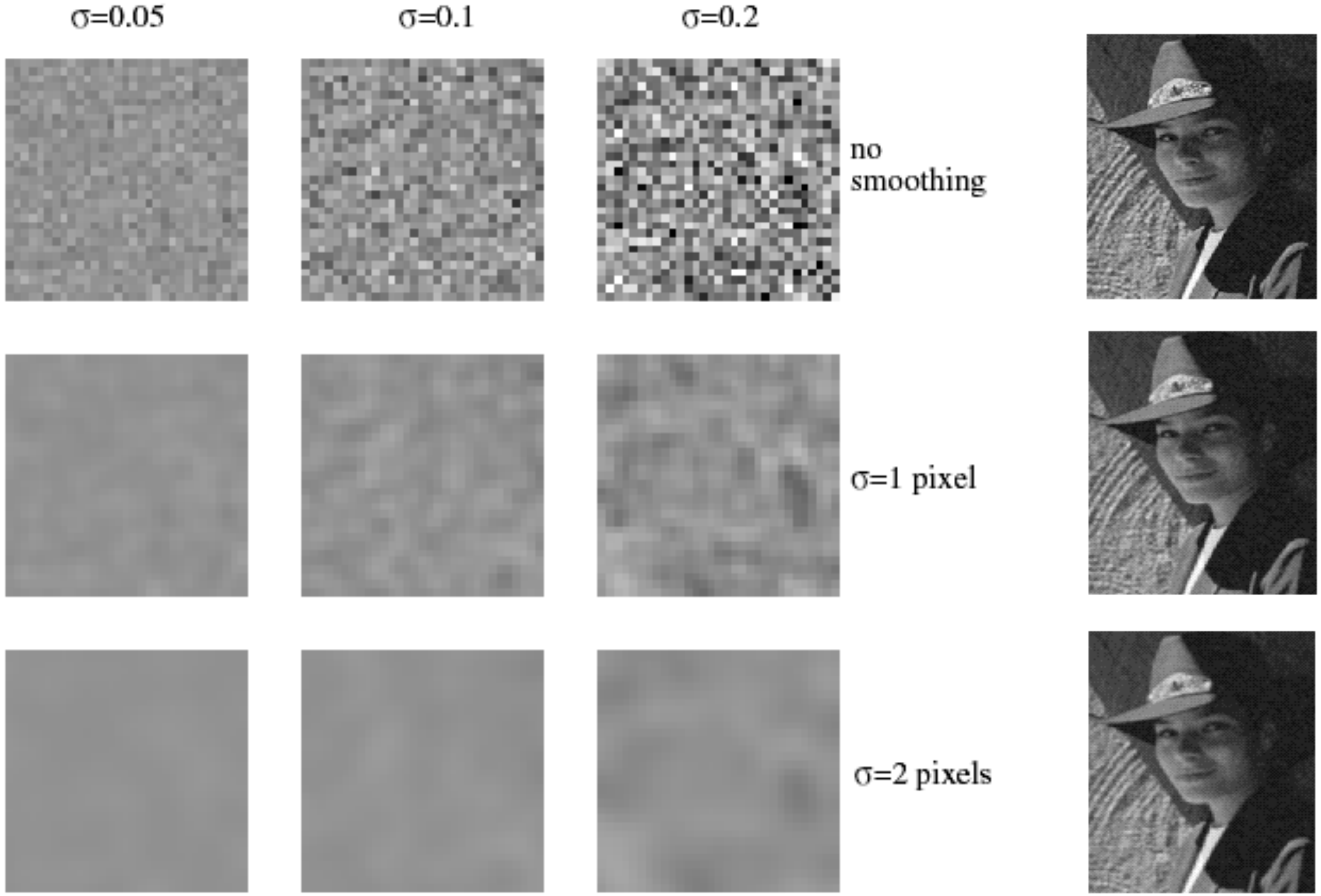


$$f(x,y) = \overbrace{\tilde{f}(x,y)}^{\text{Ideal Image}} + \overbrace{\eta(x,y)}^{\text{Noise process}}$$

Gaussian i.i.d. ("white") noise:
$\eta(x,y) \sim \mathcal{N}(\mu,\sigma)$

Source: M. Hebert

Reducing Gaussian noise



Smoothing with larger standard deviations suppresses noise, but also blurs the image

Slide from Lana Lazebnik

Reducing salt-and-pepper noise

3x3

5x5

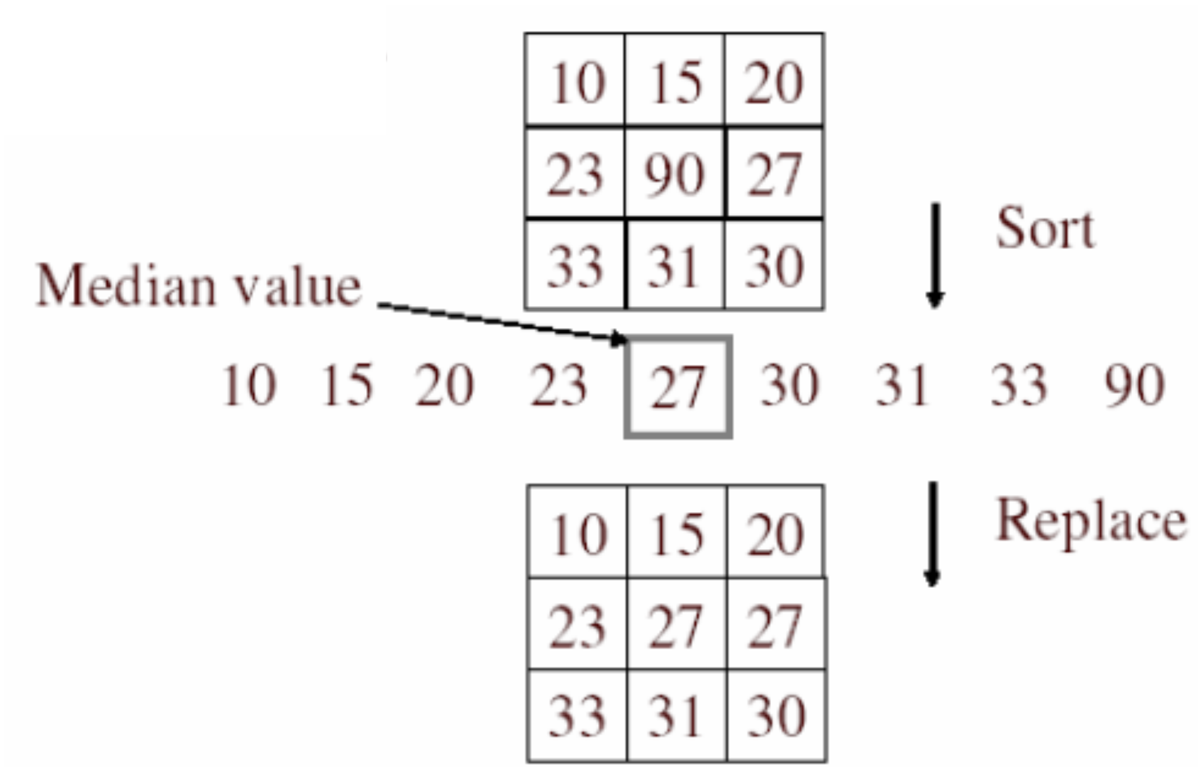
7x7

What's wrong with the results?

Slide from Lana Lazebnik

Alternative idea: Median filtering

- A **median filter** operates over a window by selecting the median intensity in the window



- Is median filtering linear?

Source: K. Grauman

Median filter

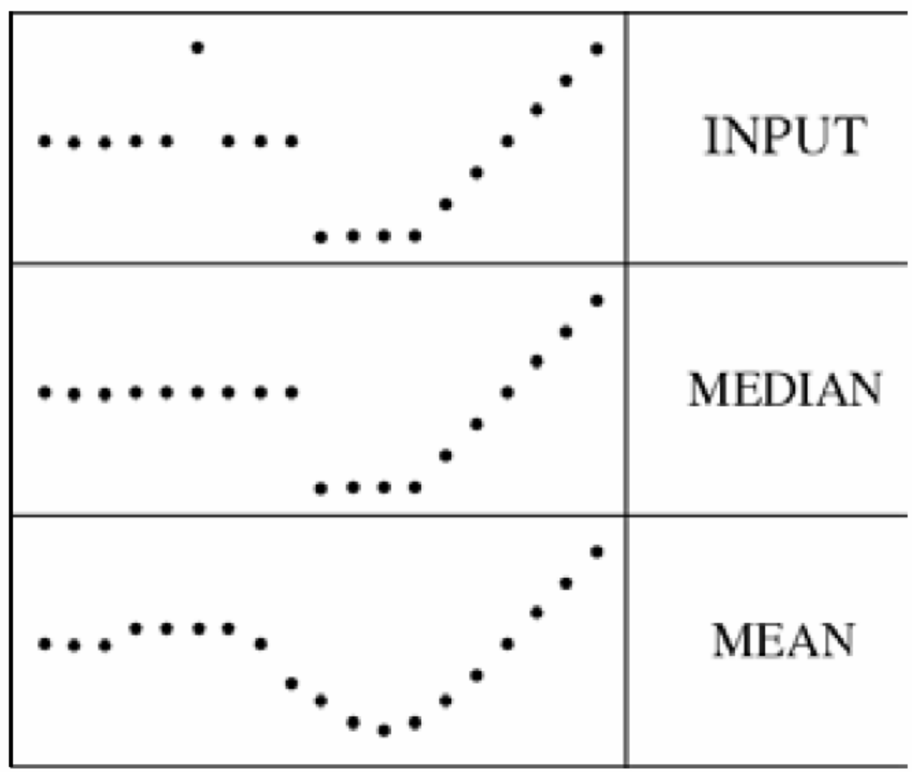
- Is median filtering linear?
- Let's try filtering

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Median filter

- What advantage does median filtering have over Gaussian filtering?
  - Robustness to outliers

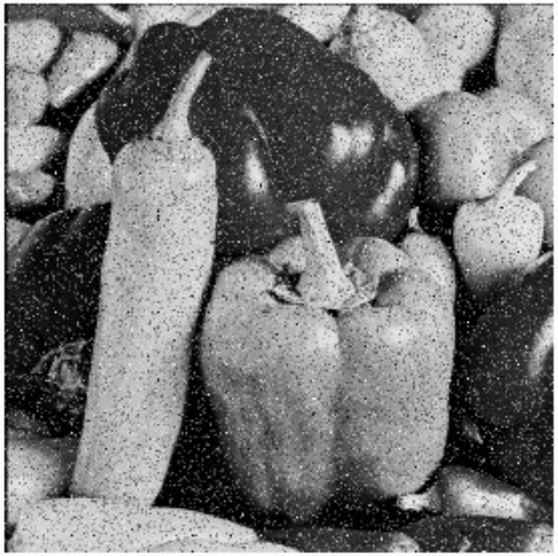
filters have width 5 :

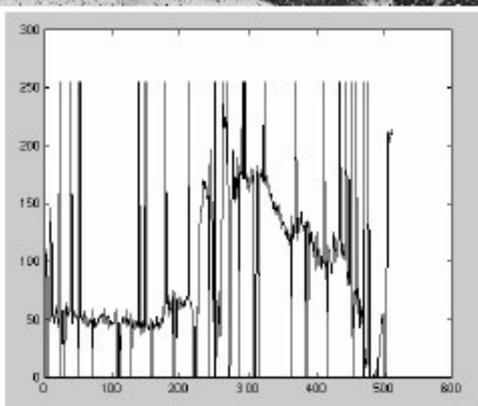


Source: K. Grauman

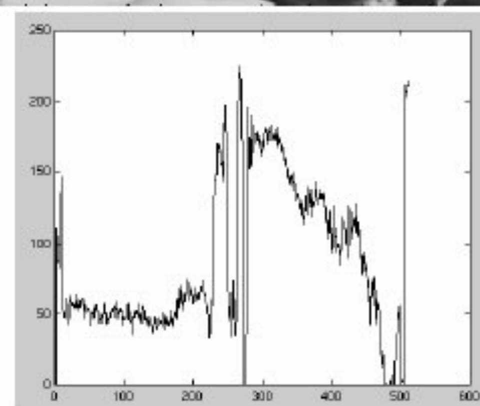
Median filter

Salt-and-pepper noise





Median filtered



Source: M. Hebert

Gaussian vs. median filtering

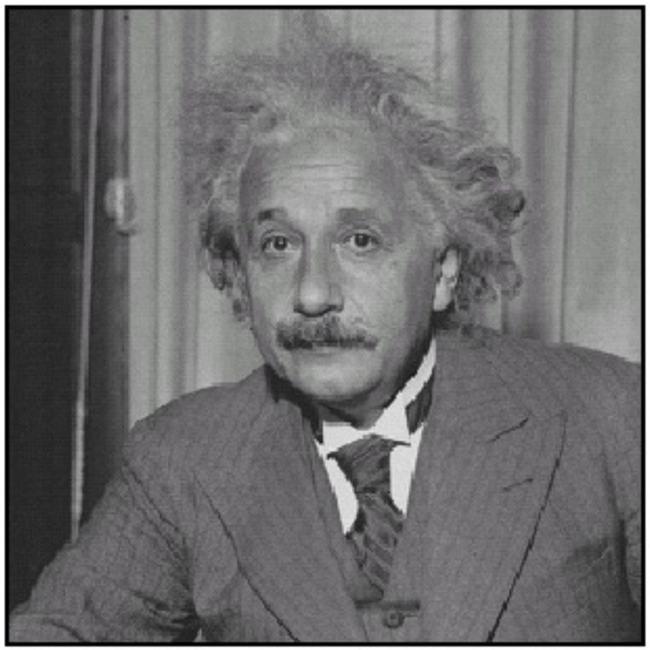
3x3 | 5x5 | 7x7

Gaussian

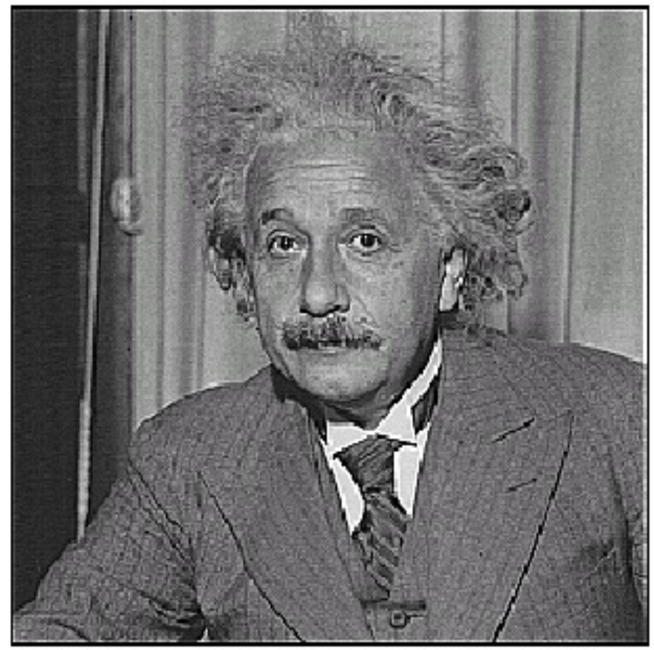
Median

Slide from Lana Lazebnik

Sharpening revisited



before



after

Source: D. Lowe

Sharpening

What does blurring take away?

–

=

Let’s add it back in.

$+\alpha$

=

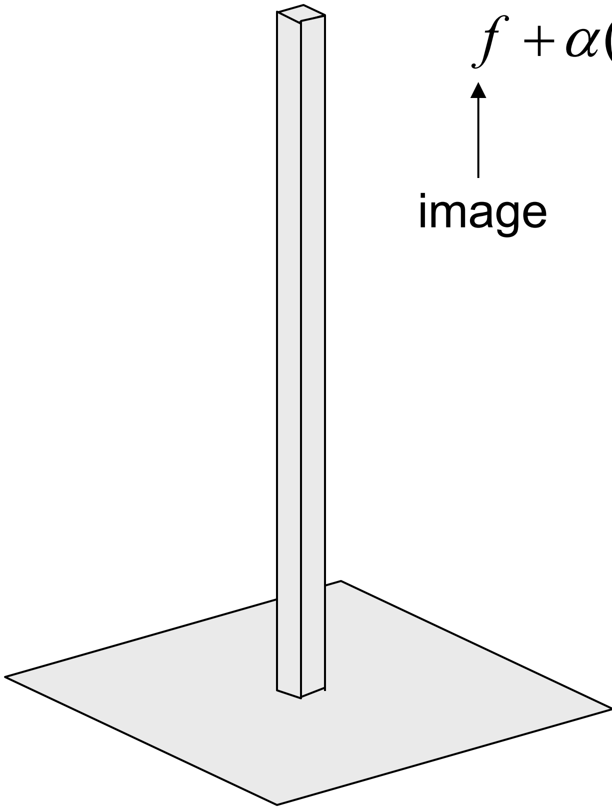
Unsharp mask filter

$$f + \alpha(f - f * g) = (1+\alpha)f - \alpha f * g = f * ((1+\alpha)e - g)$$

image

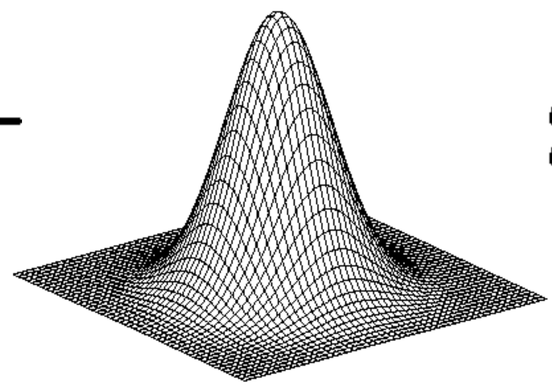
blurred image

unit impulse (identity)



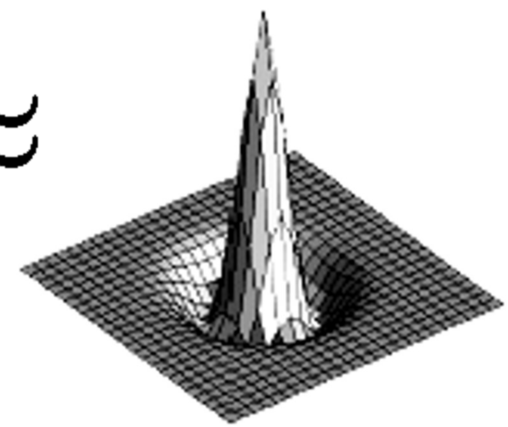
unit impulse

$-$



Gaussian

$\approx$



Laplacian of Gaussian

Next Class: Frequency view of filtering

Slide from Lana Lazebnik

Guess the filter

Guess the filter

Review: Image filtering

- Convolution
- Box vs. Gaussian filter
- Separability
- Median filter